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A class of distorted Gaussian copulas: theories and applications

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ABSTRACT
This study introduces a novel copula class, referred to as the distorted GAB copula (hereafter, dGAB copula), as an alternative to the Gaussian copula, which has shown limitations in capturing tail dependence. Much like the Gaussian copula, the dGAB copula can be uniquely determined by its bivariate marginal copulas and offers effective tail dependence modeling capabilities. To demonstrate its practical applicability, we showcase its use in the valuation of basket default swaps. Furthermore, we propose a parameter estimation approach based on the EM algorithm tailored to the dGAB copula.

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High-water mark; mean-variance framework; time consistency; transaction cost

1. Introduction

Over the past decade, the hedge fund industry has witnessed the emergence of a distinctive performance contract. Hedge fund managers commonly receive a share of annual returns exceeding the high-water mark, which represents the maximum share value since each investor's initial investment. These performance fees typically range from 15% to 25% of new profits annually, in addition to a regular annual fee of one to two percent of portfolio assets. For instance, George Soros' Quantum Fund charges a one percent annual fee based on net asset value, coupled with a 20% high-water mark performance fee on annual net new profits. Consequently, the Quantum Fund achieved a 49% (pre-fee) return in 1995 on net assets amounting to 3.7 billion, leading to an estimated total compensation of 393 million for the year. This substantial sum was primarily attributed to the incentive terms in place. However, in years where the high-water mark is not reached, manager returns see a significant reduction. In 1996, the Quantum Fund incurred a loss of 1.5%, earning only their regular annual fee of 54 million (1% of the 5.4 billion in assets).

Copulas are a useful statistical tool that enables the characterization of dependence structures among multiple random variables. They possess uniform $[0,1]$ marginal distributions, making them a crucial tool in high-dimensional statistical analysis (Joe, 1997; Nelsen, 2006). Within the field of quantitative risk management, copulas have gained widespread use due to their ability to model and mitigate tail risk, as well as their application in portfolio optimization (Denuit et al., 2006; McNeil et al., 2015).

1.1. Motivation and literature review

The Gaussian copula, which is derived from the multivariate normal distribution, is a widely utilized copula model in both theory and practice. However, as observed by Donnelly and Embrechts (2010), the Gaussian copula's inadequacy in describing tail dependence has been a subject of criticism. Tail dependence, which quantifies the co-movements of a pair of random variables in the tails of their distributions, is a crucial metric in risk management. It measures the extent to which devastating losses in portfolios or defaults of financial enterprises can occur simultaneously (Grundke & Polle, 2012; Puccetti & Rueschendorf, 2014). Hence, accurately measuring the tail dependence in copula-described bivariate random variables is of critical importance. The most popular measures of tail dependence, including upper and lower tail dependences, were introduced by Joe (1997) and a non-zero value of these measures signifies the presence of tail dependence. The Gaussian copula, however, is well-known to have both upper and lower tail dependence measures equal to zero, thereby indicating its incapacity to describe tail dependence; see also Hull and White (2004) for more information about the tail dependences.

The modification of tail dependence in a copula can be accomplished through the use of the "convex sum" approach. This approach involves the creation of a new copula, $C^{\text{convexsum}} = \alpha C_\Sigma + (1-\alpha)C$, by combining a Gaussian copula C_Σ with another copula C via the weight $\alpha \in (0,1)$. Such a new copula possesses the ability to capture tail dependence. The choice of C is critical, as it dictates the tail

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dependence behavior of the new copula $C^{\text{convexsum}}$ through its definition as a tail dependence measure. Archimedean copulas are commonly used due to their ability to model the tail dependence, but they have limited parameters and thus lack flexibility in high dimensional modeling. To address this issue, the convex sum approach becomes a search for an appropriate multivariate copula C with tail dependence properties.

In this article, we use the $C^{\mathcal{A},\mathcal{B}}$ copula proposed by Yang et al. (2009) as the copula C. This copula can be seen as a limiting case of the Gaussian copula, with the ability to be determined by its bivariate marginal copulas and exhibiting tail dependence. Mathematically, the $C^{\mathcal{A},\mathcal{B}}$ copula can be regarded as a limiting case of the Gaussian copula by noting that in the bivariate case, $\lim_{\rho\to 1^-} C_\rho(u,v) = M(u,v)$, $\lim_{\rho\to 0} C_\rho(u,v) = \Pi(u,v)$, and $\lim_{\rho\to -1^+} C_\rho(u,v) = W(u,v)$ for all $(u,v) \in [0,1]^2$, where $C_\rho(u,v)$ is a bivariate Gaussian copula with correlation coefficient ρ, $M(u,v) = \min(u,v)$ is the comonotonic copula, $\Pi(u,v) = uv$ is the independent copula, and $W(u,v) = \max(u+v-1,0)$ is the countermonotonic copula. The most prominent feature of the $C^{\mathcal{A},\mathcal{B}}$ copula resides in its similarity to the Gaussian copula, as it can be uniquely determined through its bivariate marginal copulas. This attribute engenders significant convenience, particularly in the context of high-dimensional modeling. On the other hand, a paramount distinction of the $C^{\mathcal{A},\mathcal{B}}$ copula lies in its capacity for tail dependence analysis. In contrast to alternative multivariate copulas exhibiting tail dependence, such as the student-t copula and Archimedean copula, the $C^{\mathcal{A},\mathcal{B}}$ copula exhibits heightened flexibility and manageability due to its simplicity and intuitiveness (see Section 2.1 for more details). Consequently, the $C^{\mathcal{A},\mathcal{B}}$ copula surpasses other tail-dependent copulas, including the student-t and Archimedean copulas, in terms of its enhanced adaptability and computational feasibility.

The convex sum approach, although useful, is relatively simple. A more sophisticated approach, the "distorted mix method" (DMM) proposed by Li et al. (2014), improves upon the convex sum approach by incorporating distortion functions and allowing for separate modelings of the center and tail parts of a copula. Specifically, for any given component copulas $C_1, C_2, \ldots, C_m$, the DMM implies a new family of copulas taking the form $\sum_{i=1}^m \alpha_i C_i\big(D_{i1}(u_1), D_{i2}(u_2), \ldots, D_{id}(u_d)\big)$ where D_{ij} is the distortion function satisfying $\sum_{i=1}^m \alpha_i D_{ij}(u) = u$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, d$.

By virtue of the convex sum method, the copula constructed by DMM can be arbitrarily "close" to any one of the component copulas, as long as its weight is close to one. On the other hand, the distribution function plays a critical role in DMM for the modification of the tail dependence. As demonstrated by Li et al. (2014), the copula constructed by DMM can model dependence structure by handling the central and tail parts separately. More importantly, the tail dependence of a given copula can be modified to any desired pattern. In addition to the tail dependence modeling, the distortion function also plays a key role in the unique determination of dGAB copula by its marginals (see Theorem 3.2). Obviously, the copula constructed by DMM will degenerate to the convex sum approach when trivial distortion functions are used. In this article, we demonstrate the flexibility of DMM by constructing a new family of copulas based on the Gaussian copula and $C^{\mathcal{A},\mathcal{B}}$ copula.

1.2. Main contribution

This article introduces a new family of copulas called the" dGAB copula" by utilizing the DMM approach, which is based on two component copulas: the Gaussian copula and the $C^{\mathcal{A},\mathcal{B}}$ copula. We systematically analyze the properties of the dGAB copula and specifically highlight its advantages. (1) Uniqueness: The most prominent feature presented in this article is that the dGAB copula can be uniquely determined by its bivariate marginal copulas. This provides a more convenient solution for high-dimensional statistical modeling compared to Li et al. (2014). (2) Flexibility: The dGAB copula inherits advantageous attributes from the DMM framework. Notably, it has the capacity to approach arbitrary levels of proximity to the Gaussian copula. This attribute assures that the dGAB copula exhibits statistical properties similar to those of the Gaussian copula. (3) Explicit Probability Structure: The dGAB copula exhibits an explicit probability structure, making it more tractable in the context of Monte Carlo simulations and enhancing its utility in multidimensional modeling. (4) Enhanced Tail Dependence: The dGAB copula demonstrates enhanced tail dependence performance across multiple dimensions, improving both lower and upper tail dependence as well as upper-left and lower-right tail dependence. (5) Application: We apply the dGAB copula to the basket default pricing problem, enabling a more refined correlation structure among default times and obtaining closed-form solutions. Additionally, the dGAB copula has versatile applications in operations research, including the newsvendor model (Alwan et al., 2016; Chen et al., 2009).

In the practical investigation, we propose the dGAB-EM algorithm for the parameter estimation.

The findings indicate that the dGAB copula can serve as an appropriate alternative when a single Gaussian copula fails to model the tail dependence of multiple random variables.

This article is organized as follows: In Section 2, two families of copulas are introduced, derived from the Gaussian copula and the $C^{\mathcal{A},\mathcal{B}}$ copula. The main results of this study are developed in Section 3. The examination of tail dependence and the probability structure are discussed in Section 4. Applications to the BDS (basket default swap) and empirical studies are presented in Section 5. Finally, the research is concluded in Section 6. The accompanying online appendices provide the mathematical proofs for all relevant results.

2. The model

2.1. Preliminaries about the $C^{\mathcal{A},\mathcal{B}}$ copula

In the introduction section, the concept of $C^{\mathcal{A},\mathcal{B}}$ copula as a component copula is introduced. The $C^{\mathcal{A},\mathcal{B}}$ copula is defined as the joint distribution of uniform $[0,1]$ random variables $U_1, U_2, \ldots, U_n$ if the following two assumptions are satisfied:

Assumption A. *There exists a uniform* $[0,1]$ *random variable* U *such that random variables* $U_1, U_2, \ldots, U_n$ *are conditionally independent given the common factor* U.

Assumption B. *For each* $i = 1, 2, \ldots, n$, *the joint distribution of* U_i *and* U *is bivariate Fréchet, with the copula expressed as* $C_i(u,v) = a_{i1}M(u,v) + a_{i2}\Pi(u,v) + a_{i3}W(u,v)$, *where* $a_{ij} \geq 0$ *and* $a_{i1} + a_{i2} + a_{i3} = 1$ *for* $j = 1, 2, 3$.

With the above specifical settings, Yang et al. (2009) demonstrates that the joint distribution of $U_1, U_2, \ldots, U_n$ takes the following form:

$$C^{\mathcal{A},\mathcal{B}}(u_1, u_2, \ldots, u_n) = \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{ij_i} \right) W\left(\min_{i \leq n, j_i=1} \{u_i\}, \min_{i \leq n, j_i=3} \{u_i\} \right) \prod_{i \leq n, j_i=2} u_i, \tag{1}$$

which is called the $C^{\mathcal{A},\mathcal{B}}$ copula. Notably, one of the most remarkable features of the $C^{\mathcal{A},\mathcal{B}}$ copula is that it can be uniquely determined by its bivariate marginal copulas.

By Equation (1), the bivariate marginal copulas are all Fréchet copulas, which can be expressed as

$$C^{\mathrm{F}}_{d_{ij}^{+}, d_{ij}^{-}}(u,v) = d_{ij}^{+} M(u,v) + d_{ij}^{\perp} \Pi(u,v) + d_{ij}^{-} W(u,v), \tag{2}$$

where $d_{ij}^{+} = a_{i1}a_{j1} + a_{i3}a_{j3}, d_{ij}^{-} = a_{i1}a_{j3} + a_{i3}a_{j1}$, and $d_{ij}^{\perp} = 1 - d_{ij}^{+} - d_{ij}^{-}$, for $i \neq j \in \{1, 2, \ldots, n\}$.

Notably, the comonotonic and counter-monotonic copulas in Fréchet copula (2) are typically referred to as the Fréchet-Hoeffding bounds, which are the pointwise upper and lower bounds for any bivariate copulas, i.e. $W(u,v) \leq C(u,v) \leq M(u,v)$ holds for any bivariate copula C. Additionally, it is noteworthy that the three constituent copulas within the Fréchet copula correspond to three different extreme dependence structures: perfect positive dependence, independence, and perfect negative dependence. Furthermore, it is possible to approximate any general copula through a convex combination of these component copulas, as demonstrated by Yang et al. (2006).

It should be emphasized that Assumption B is relatively simple for practical modeling, as it is merely a mixture of three copulas of extreme correlations, and two of them are even singular ones. In the pursuit of enhancing both tractability and adaptability, we will undertake a refinement of the original Assumptions A and B in this article (see Theorem 4.3), thereby yielding a version that better aligns with practical modeling considerations.

2.2. The GAB copula

As noted in the introduction, we introduce a new family of copulas by the convex sum approach. This approach involves the combination of two component copulas, C_1 and C_2, with a weight parameter $\alpha \in (0,1)$. In this study, the Gaussian copula is selected as C_1, and $C^{\mathcal{A},\mathcal{B}}$ copula is selected as C_2. Therefore, a new family of copula, denoted by $C^{\mathcal{A},\mathcal{B}}_{\alpha,\Sigma}$, is obtained.

Definition 1 (GAB copula). *For any constant* $\alpha \in (0,1)$,

$$\mathrm{GAB}(\mathbf{u}) = \alpha C_{\Sigma}(\mathbf{u}) + (1-\alpha) C^{\mathcal{A},\mathcal{B}}(\mathbf{u}), \mathbf{u} \in [0,1]^n. \tag{3}$$

is called the GAB copula.

Incorporating the distortion functions, we extend the GAB copula to the distorted GAB copula in the next section.

2.3. The dGAB copula

The DMM framework proposes a novel class of copulas by incorporating distortion functions. For technical reasons, the definition of the distortion function is restricted to a specific class of strictly increasing and differentiable functions. Formally, we have the following definition.

Definition 2 (Distortion function). *A function D is called a distortion function if it is differentiable and strictly increasing from* $[0,1] \to [0,1]$ *with* $D(0) = 0$ *and* $D(1) = 1$. *Moreover, a distortion function D is called trivial if* $D(u) = u$ *for all* $u \in [0,1]$.

The second column in Table 1 provides more examples of commonly used distortion functions.

To define the dGAB copula, the distortion functions must satisfy "Assumption A" in Li et al. (2014). For the sake of clarity and brevity, we shall henceforth denote this assumption as the "feasibility" criterion, which is given in the following definition.

Definition 3 (Feasible pair). *Two distortion functions (D, E) is called a feasible pair if there exists a constant* $\alpha \in (0,1)$ *such that* $\alpha D(u) + (1-\alpha)E(u) = u$ *holds for all* $u \in [0,1]$.

The feasibility of distortion functions is necessary to ensure that the dGAB copula is well-defined. For the distortion functions in Table 1, the third column presents their feasible pairs.

For a feasible pair of distortion functions (D, E), it is important to emphasize that $\alpha D'(u) + (1-\alpha)E'(u) = 1$ holds true for all $u \in [0,1]$. Consequently, we can deduce that $D'(u)$ falls within the range $(0, 1/\alpha]$, and $E'(u)$ is bounded within $\left(0, 1/(1-\alpha)\right]$ for all $u \in [0,1]$. In other words, it is necessary that α satisfies $\alpha \in \left(0, 1/D'(u)\right]$ and $\alpha \in \left[1 - 1/E'(u), 1\right)$ for all $u \in [0,1]$. These conditions can be viewed as hidden constraints for the distortion function pair (see the last column in Table 1).

By incorporating feasible pairs of distortion functions, the DMM defines a new family of copulas based on the Gaussian copula and $C^{\mathcal{A},\mathcal{B}}$ copula, referred to as the "dGAB" copulas in this article.

Definition 4 (dGAB copula). *Let* (D_i, E_i) *be a feasible pair of distribution functions for* $i = 1, 2, \ldots, n$, *then for any constant* $\alpha \in (0,1)$,

$$\mathrm{dGAB}(\mathbf{u}) = \alpha C_\Sigma(\mathbf{D}(\mathbf{u})) + (1-\alpha)C^{\mathcal{A},\mathcal{B}}(\mathbf{E}(\mathbf{u})), \mathbf{u} \in [0,1]^n, \tag{4}$$

is called the dGAB copula, where $\mathbf{D} = (D_1, D_2, \ldots, D_n)$ *and* $\mathbf{E} = (E_1, E_2, \ldots, E_n)$.

It is straightforward to see that the bivariate marginal of the dGAB copula takes the following form:

$$\mathrm{dGF}(u,v) = \alpha C_{\rho_{ij}}\left(D_i(u), D_j(v)\right) + (1-\alpha)C^{\mathrm{F}}_{d^+_{ij}, d^-_{ij}}\left(E_i(u), E_j(v)\right), \tag{5}$$

where $d^+_{ij} = a_{i1}a_{j1} + a_{i3}a_{j3}, d^-_{ij} = a_{i1}a_{j3} + a_{i3}a_{j1}$, and $d^\perp_{ij} = 1 - d^+_{ij} - d^-_{ij}$. The copula dGF in Equation (5) will be called the 'distorted Gaussian-Fréchet copula' in the subsequent developments.

Obviously, when the two families of distortion functions are trivial, the dGAB copula degenerates to the GAB copula. Accordingly, the bivariate marginal copulas of GAB copula degenerate to the "GF copula" as $\mathrm{GF}(u,v) = \alpha C_{\rho_{ij}}(u,v) + (1-\alpha)\ C^{\mathrm{F}}_{d^+_{ij}, d^-_{ij}}(u,v)$.

3. Unique determination by bivariate marginals

In this section, we will prove that dGAB (GAB) copulas can be uniquely determined by their bivariate marginal copulas, which is one of the most prominent properties of the dGAB copula.

3.1. dGAB copula: Unique determination by bivariate marginal copulas

In the examination of the property of unique determination through bivariate marginal copulas, we differentiate between two cases: $\Sigma = I_n$ and $\Sigma \neq I_n$, where I_n is the identity matrix. This distinction is crucial as it accounts for the potential ambiguity in the weight of the independent copula resulting from varying linear coefficient matrices.

3.1.1. Unique determination when $\Sigma \neq I_n$

The following theorem shows that the dGAB with $\Sigma \neq I_n$ copula can be uniquely determined by its bivariate marginal copulas.

Theorem 3.1 (dGAB copula with $\Sigma \neq I_n$). *For any given correlation matrix* $\Sigma \neq I_n$, *the corresponding dGAB copula, denoted as* $\mathrm{dGAB}(\mathbf{u}) = \alpha C_\Sigma(\mathbf{D}(\mathbf{u})) + (1-\alpha)C^{\mathcal{A},\mathcal{B}}(\mathbf{E}(\mathbf{u}))$, *can be uniquely determined by its bivariate marginal copulas.*

It was previously noted that the GAB copula is a specific instance of the dGAB copula, in which all

Table 1. The second column presents three examples of distortion functions and the third column presents the corresponding feasible pairs. Note that $0 < \alpha < 1$ is the weight of the Gaussian copula in the dGAB or GAB copulas.

Distortion function	$D_i(u)$	$E_i(u)$	Condition
Power	$D_1(u) = u^\beta, \beta > 0$	$E_1(u) = \frac{u - \alpha u^\beta}{1-\alpha}$	$\alpha\beta \leq 1$
Li et al. (2014)	$D_2(u) = \frac{\alpha u^2}{1-\alpha+(2\alpha-1)u}$	$E_2(u) = \frac{u-(1-\alpha)u^2}{1-\alpha+(2\alpha-1)u}$	$0 < \alpha < 1$
Exponential	$D_3(u) = \frac{e^{\beta u}-1}{e^\beta - 1}, \beta > 0$	$E_3(u) = \frac{(e^\beta-1)u-\alpha(e^{\beta u}-1)}{(1-\alpha)(e^\beta-1)}$	$\alpha \leq \frac{1-e^{-\beta}}{\beta}$

distortion functions are trivial. Consequently, the theorem above is also applicable to the GAB copula. Therefore, we have the following result.

Corollary 3.1 (GAB copula with $\Sigma \neq I_n$). *For any given correlation matrix $\Sigma \neq I_n$, the corresponding GAB copula, denoted as $\mathrm{GAB}(\mathbf{u}) = \alpha C_\Sigma(\mathbf{u}) + (1-\alpha)C^{\mathcal{A},\mathcal{B}}(\mathbf{u})$, can be uniquely determined by its bivariate marginal copulas.*

The dGAB copula, defined as a mixture of Gaussian and $C^{\mathcal{A},\mathcal{B}}$ copulas with distortion functions, has a straightforward structure. Despite its simplicity, our results demonstrate that the dGAB copula possesses inherited advantageous properties from both Gaussian and $C^{\mathcal{A},\mathcal{B}}$ copulas. Furthermore, as noted in Li et al. (2014), the dGAB copula can approach the Gaussian copula arbitrarily closely when the weight α approaches to 1. This is quantified by the following inequality:

$$\int_{[0,1]^n} \left|\mathrm{dGAB}(\mathbf{u}) - C_\Sigma(\mathbf{u})\right| \, \mathrm{d}\mathbf{u} \leq \min\left\{1, (1-\alpha)\left(\frac{n}{2}+1\right)\right\},$$

Such proximity between the dGAB copula and Gaussian copula, including the special case of the GAB copula, confers a desirable level of statistical tractability.

3.1.2. Unique determination when $\Sigma = I_n$

When $\Sigma = I_n$, the dGAB copula cannot be determined by its bivariate marginal copulas in general. Additional conditions are required.

Theorem 3.2 ($\Sigma = I_n$). *If the correlation matrix $\Sigma = I_n$ and there are at least two pairs of non-trivial distortion functions, the corresponding dGAB copula, denoted as $\mathrm{dGAB}(\mathbf{u}) = \alpha C_\Sigma(\mathbf{D}(\mathbf{u})) + (1-\alpha)C^{\mathcal{A},\mathcal{B}}(\mathbf{E}(\mathbf{u}))$, can be uniquely determined by its bivariate marginal copulas.*

The preceding theorem highlights that the dGAB copula, when the component Gaussian copula degenerates to the independent copula, may not possess the property of unique determination through its bivariate marginal copulas. However, with the additional requirement of the existence of at least two pairs of non-trivial distortion functions, the dGAB copula can indeed be uniquely determined. It is important to note that these conditions are merely sufficient and not necessary, as evidenced by the proof of the theorem.

Moreover, the theorem also indicates the possibility that the GAB copula with $\Sigma = I_n$ may not be uniquely determined by its bivariate marginal copulas. This issue will be further investigated in the subsequent section.

3.1.3. Existence of dGAB copula under given bivariate marginals

Given a family of bivariate distorted Gaussian-Fréchet copulas, a natural problem is whether there exists a dGAB copula with the given family of copulas as its bivariate marginal copulas.

Theorem 3.3. *Given bivariate distorted Gaussian-Fréchet marginal copulas dGF_{im} for $1 \leq i < m \leq n$, namely*

$$\mathrm{dGF}_{im}(u,v) = \alpha_{im} C_{\rho_{im}}(D_i(u), D_m(v)) + (1-\alpha_{im}) C^{\mathrm{F}}_{d^+_{im}, d^-_{im}}(E_i(u), E_m(v)),$$

where the constants d^+_{im} and d^-_{im} are non-negative satisfying $d^+_{im} + d^-_{im} \leq 1$. There exist uniform $[0,1]$ random variables U_i, $i = 1, 2, \ldots, n$ following the dGAB copula such that $\mathbb{P}(U_i \leq u, U_m \leq v) = \mathrm{dGF}_{im}(u,v)$ if and only if there exist non-negative constants $\alpha \in (0,1)$ and a_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, 3$ satisfying that $\sum_{j=1}^{3} a_{ij} = 1$ such that

$$d^+_{im} = a_{i1}a_{m1} + a_{i3}a_{m3} \quad \text{and} \quad d^-_{im} = a_{i1}a_{m3} + a_{i3}a_{m1}. \tag{6}$$

The above theorem gives a necessary and sufficient condition for the existence of the dGAB copula under given bivariate distorted Gaussian-Fréchet copulas. However, the uniqueness cannot be established due to the counterexample in the subsequent section.

3.2. A Counterexample

Regrettably, the property of unique determination through bivariate marginal copulas is not satisfied by the GAB copulas with $\Sigma = I_n$. This is formalized by the following proposition.

Proposition 3.1. *If the correlation matrix $\Sigma = I_n$, then the corresponding GAB copula, denoted as $\mathrm{GAB}(\mathbf{u}) = \alpha C_\Sigma(\mathbf{u}) + (1-\alpha)C^{\mathcal{A},\mathcal{B}}(\mathbf{u})$, cannot be uniquely determined by its bivariate marginal copulas.*

It has been established through the direct calculation that the bivariate marginal copulas of the GAB copula with $\Sigma = I_n$ are Fréchet copulas. This class of copulas takes the form $\xi_1 M(u,v) + \xi_2 \Pi(u,v) + \xi_3 W(u,v)$, where the coefficients ξ_1, ξ_2, and ξ_3 are non-negative and satisfy $\xi_1 + \xi_2 + \xi_3 = 1$. It is worth noting that the Fréchet copula can be represented as a convex combination of an independent copula and another Fréchet copula. This leads to a situation where the weight of the independent copula in the bivariate marginal copula is the sum of the weights of the Gaussian copula and $C^{\mathcal{A},\mathcal{B}}$ copula, rendering the weights of the Gaussian copula and $C^{\mathcal{A},\mathcal{B}}$ copula indistinguishable.

4. Tail dependence and probability structure

4.1. Tail dependencies

As mentioned in the introduction section, the Gaussian copula has faced criticisms for its inability to accurately quantify tail dependence among marginal risks (Bassamboo et al., 2008; Jin et al., 2021; Wang and Dyer, 2012). In comparison, a significant advantage of the dGAB copula is its improved capability in capturing tail dependence. The conventional definition of tail dependence can be characterized by the lower and upper tail dependence coefficients (Joe, 1997), denoted as λ_L and λ_U, which are defined as follows:

$$\lambda_L(X,Y) = \lim_{u\to 0^+} \mathbb{P}\big(X \le F_X^{-1}(u) | Y \le F_Y^{-1}(u)\big) = \lim_{u\to 0+} \frac{C(u,u)}{u}, \tag{7}$$

$$\lambda_U(X,Y) = \lim_{u\to 1^-} \mathbb{P}\big(X > F_X^{-1}(u) | Y > F_Y^{-1}(u)\big) = \lim_{u\to 1-} \frac{1-2u+C(u,u)}{1-u}, \tag{8}$$

respectively, where C represents the copula of X and Y.

However, it is imperative to acknowledge that the concept of tail dependence primarily applies to positively correlated random variables, as both lower and upper tail dependence coefficients are designed to measure their comovements in the tails of their distributions. In contrast, when dealing with negatively correlated random variables (e.g. the crude oil price and airline stock price), a new definition of tail dependence becomes necessary to adequately capture the behavior in the tail dependence.

Definition 5 *(Sub-tail dependence). The sub-tail lower and upper-tail dependence coefficients of two random variables X and Y are defined by*

$$\lambda_L^{\text{sub}}(X,Y) = \lim_{u\to 0+} \mathbb{P}\big(X \le F_X^{-1}(u) | Y > F_Y^{-1}(1-u)\big) = \lim_{u\to 0+} \frac{u-C(u,1-u)}{u}, \tag{9}$$

$$\lambda_U^{\text{sub}}(X,Y) = \lim_{u\to 0+} \mathbb{P}\big(X > F_X^{-1}(1-u) | Y \le F_Y^{-1}(u)\big) = \lim_{u\to 0+} \frac{u-C(1-u,u)}{u}, \tag{10}$$

respectively, where C represents the copula of X and Y.

It is essential to highlight that some conventional tail-dependent copulas, such as Archimedean copulas (e.g. Clayton and Gumbel copulas), are inherently limited in their capacity to capture sub-tail dependencies. In contrast, the dGAB copula stands out as an advantageous choice due to its unique capability to not only characterize tail dependencies but also sub-tail dependencies by Equations (9) and (10). This distinctive feature renders the dGAB copula a useful tool in the realm of statistical modeling, particularly when confronted with scenarios involving multiple random variables exhibiting concurrent positive and negative correlations.

Theorem 4.1. *Suppose that the distortion functions in the dGAB are all identical[1], i.e. $D_i = D$ and $E_i = E$ for $i = 1,2,\dots,n$. Then the tail dependence coefficients of the bivariate marginal copulas are given by*

$$\lambda_L\big(\text{dGAB}_{ij}\big) = (1-\alpha)(a_{i1}a_{j1}+a_{i3}a_{j3})E'(0+),$$
$$\lambda_U\big(\text{dGAB}_{ij}\big) = (1-\alpha)(a_{i1}a_{j1}+a_{i3}a_{j3})E'(1-),$$
$$\lambda_L^{\text{sub}}\big(\text{dGAB}_{ij}\big) = (1-\alpha)\big(1-\big(1-(a_{i1}a_{j3}+a_{i3}a_{j1})E'(0+)\big)\big),$$
$$\lambda_U^{\text{sub}}\big(\text{dGAB}_{ij}\big) = (1-\alpha)\big(1-\big(1-(a_{i1}a_{j3}+a_{i3}a_{j1})E'(0+)\big)\big),$$

respectively, where dGAB_{ij} is the (i, j)-dimensional marginal copula of the dGAB copula with different indices $i,j = 1,2,\dots,n$.

This theorem suggests that the dGAB copula's capability of exhibiting tail dependence is solely attributed to the presence of the $C^{\mathcal{A},\mathcal{B}}$ copula and the related distortion functions. As a direct application of this theorem, the tail dependence coefficients of the GAB copula can be obtained as follows.

Corollary 4.1. *Tail dependence coefficients of the GAB copula are given by*

$$\lambda_L(\text{GAB}_{ij}) = \lambda_U(\text{GAB}_{ij}) = (1-\alpha)(a_{i1}a_{j1}+a_{i3}a_{j3}),$$
$$\lambda_L^{\text{sub}}(\text{GAB}_{ij}) = \lambda_U^{\text{sub}}(\text{GAB}_{ij}) = (1-\alpha)(a_{i1}a_{j3}+a_{i3}a_{j1}),$$

where GAB_{ij} is the bivariate marginal copula for indices $i \neq j \in \{1,2,\dots,n\}$.

In contrast to the classical Gaussian copula, the dGAB (including GAB) copula demonstrates a distinct capacity to capture both tail and sub-tail dependencies. This phenomenon is visually evident in Figure 1, where the scatter plot of the GAB copula with standard normal marginals, displayed in the second panel, prominently reveals a greater dispersion of data points across all four corners in comparison to the scatter plot of the Gaussian copula with standard normal marginals, presented in the first panel. More importantly, incorporating the distortion functions, the dGAB copula exhibits the improved ability to capture different tail dependence patterns and show a high level of flexibility in tail dependence modeling.

[1]The assumption of "identical" is to ensure the existence of the tail dependence coefficients. Noting that $M(E_i(u),E_j(u)) = (E_i(u)+E_j(u)-|E_i(u)-E_j(u)|)/2$, the limit $\lim_{u\to 0+} M(E_i(u),E_i(u))/u$ may not exist due to the absolute value term. Therefore, we assume identical distortion functions for technical considerations.

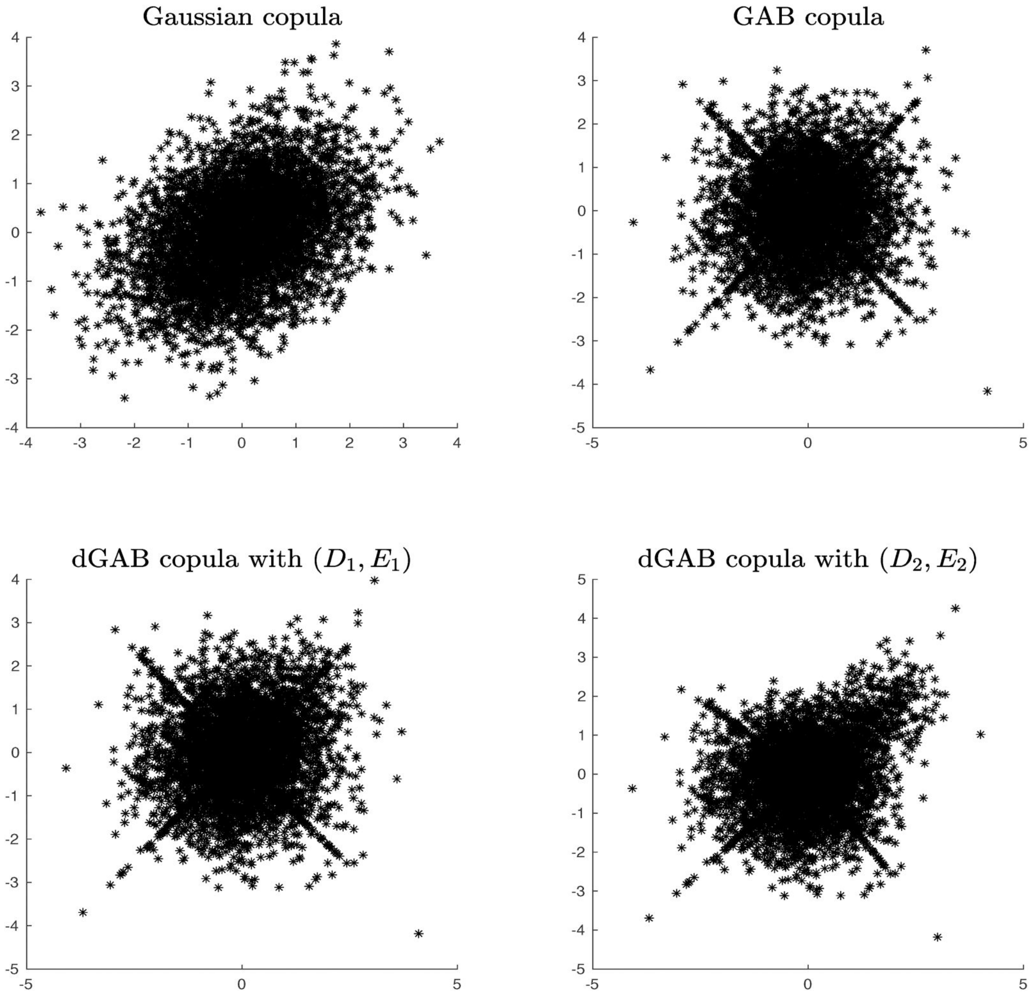

Figure 1. The top-left panel presents a scatter plot of the Gaussian copula with standard normal marginals, characterized by a correlation coefficient $\rho = 0.6$. The top-right panel presents the scatter plot of the GAB copula with the same ρ value and a weight parameter $\alpha = 0.8$ for the component Gaussian copula. The bottom-left panel presents a scatter plot of the dGAB copula using distortion functions (D_1, E_1) with $\beta = 3$, while maintaining the same ρ and $\alpha = 0.3$. Finally, the bottom-right panel presents the scatter plot of the dGAB copula with distortion functions (D_2, E_2), with $\alpha = 0.3$ and $\rho = 0.6$.

The selection of distortion functions has a crucial influence on the tail dependencies of the dGAB copula. Unlike the tail dependence coefficients of the GAB copula, the tail dependence coefficients of the dGAB copula, as shown in Theorem 4.1, incorporate two extra terms, namely $E'(0+)$ and $E'(1-)$. These two terms augment the adaptability of the dGAB copula with respect to tail dependence, thereby providing a wider scope for modeling prospects. It is imperative to note that, in accordance with the tail dependency definition, their determination relies exclusively on the characteristics of the distortion functions at the two boundary points, namely, 0 and 1. As illustrated by the last two lines in Table 2, although with two different distortion functions, their lower tail dependence coefficients remain the same. This phenomenon is attributable to the equality of their respective limits, denoted as $E_1'(0+)$ and $E_2'(0+)$, which converge to the same value of $1/(1-\alpha)$.

The third and fourth panels illustrate scatter plots for dGAB copulas with distinct pairs of distortion functions. Remarkably, these plots reveal disparate patterns of tail dependencies, underscoring the profound impact of distortion function selection on the copula's behavior in capturing tail dependence structures.

For illustration purposes, we provide the tail dependence coefficients for these four panels in Figure 1 in the table below.

Table 2. Tail dependence coefficients of the Gaussian, GAB, and dGAB copulas.

Copula	λ_L	λ_U	λ_L^{sub}	λ_U^{sub}
Gaussian	0	0	0	0
GAB	0.08	0.08	0.10	0.10
dGAB with (D_1, E_1)	0.09	0.06	0.11	0.11
dGAB with (D_2, E_2)	0.09	0	0.07	0.07

4.2. Probability structure and interpretation of the dGAB copula

In this section, we investigate the probability structure of the dGAB copula. This pertains to the explicit expressions governing random variables under this copula, with notable applications in areas such as sampling theory (see, e.g. the scatter plot in Figure 1). This article focuses on highlighting the explicit nature of the probability structure of the dGAB copula. We first give a definition of the probability structure of a copula.

Definition 6 (Probability structure). *A copula C is said to have a probability structure if there exists a random vector $(U_1, U_2, \ldots, U_n)$ with each U_i following a uniform $[0,1]$ distribution, such that $(U_1, U_2, \ldots, U_n)$ has the distribution function C, denoted as $(U_1, U_2, \ldots, U_n) \sim C$.*

In line with our introduction, the widely employed convex sum approach is a typical approach to constructing new copulas. The question naturally arises: does a copula constructed using this method maintain a probability structure? The subsequent theorem affirms that when both component copulas have probability structures, the copula constructed by the convex sum approach also retains a probability structure.

Theorem 4.2. *Consider two random vectors $(U_1, U_2, \ldots, U_n) \sim C_1$ and $(V_1, V_2, \ldots, V_n) \sim C_2$, along with a random variable $I \sim \text{Bernoulli}(\alpha)$ that is independent of all U_i and V_i. Then for any feasible pairs of distortion functions D_i and E_i for $i = 1, 2, \ldots, n$, the following holds:*

$$\mathbf{T} \sim \alpha C_1(\mathbf{D}(\mathbf{u})) + (1-\alpha)C_2(\mathbf{E}(\mathbf{u})),$$

where $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ is a random vector with $T_i = I \cdot D_i^{-1}(U_i) + (1-I) \cdot E_i^{-1}(V_i)$ for $i = 1, 2, \ldots, n$.

Notably, the above theorem gives the probability structure of the dGAB copula since both the Gaussian and $C^{\mathcal{A},\mathcal{B}}$ copulas have probability structures. As demonstrated by Yang et al. (2009), consider the collection of independent uniform $[0,1]$ random variables W, V_i, and the random partition $(A_i^+, A_i^-, A_i^\perp)$ of the underlying probability space with probabilities $\mathbb{P}(A_i^+) = a_{i1}, \mathbb{P}(A_i^-) = a_{i3}$, and $\mathbb{P}(A_i^\perp) = a_{i2}$, respectively. Moreover, the independence between W, V_i, $(A_i^+, A_i^-, A_i^\perp)$ is assumed. For each $i = 1, 2, \ldots, n$, the resulting random variables W_i defined by

$$W_i = W I_{A_i^+} + V_i I_{A_i^\perp} + (1-W) I_{A_i^-}, \tag{11}$$

follow the $C^{\mathcal{A},\mathcal{B}}$ copula, namely $(W_1, W_2, \ldots, W_n) \sim C^{\mathcal{A},\mathcal{B}}$.

On the other hand, it is well known that the Gaussian copula admits an explicit probability structure. As a result, it can be concluded that the dGAB copula also has a probability structure, which is presented in the following example.

Example 1. (Probability structure of the dGAB copula). *The marginals of the dGAB copula have the following probability structure:*

$$U_i = I \cdot D_i^{-1}\left(\Phi\left(\sum_{j=1}^{n} \delta_{ij} X_m\right)\right) + (1-I)$$
$$\cdot E_i^{-1}\left(W 1_{A_i^+} + V_i 1_{A_i^\perp} + (1-W) 1_{A_i^-}\right), \tag{12}$$

for $i = 1, 2, \ldots, n$, where

1. *$I \sim \text{Bernoulli}(\alpha)$ is a random variable independent of the remaining random variables;*
2. *$X_1, X_2, \ldots, X_n$ are standard normal random variables;*
3. *The matrix $A = (\delta_{im})_{im}$ is the result of the Cholesky decomposition, i.e. $\Sigma = A \cdot A^\top$;*
4. *W, V_i, and $(A_i^+, A_i^-, A_i^\perp), i = 1, 2, \ldots, n$, are defined by Equation (11).*

As demonstrated by Yang et al. (2009), the $C^{\mathcal{A},\mathcal{B}}$ copula has practical implications based on two assumptions. Concretely, Assumption A, as expounded within their article, delineates conditional independence based on a common factor. On the other hand, Assumption B elucidates the joint distributions characterizing the interrelationship between individual random variables and the aforementioned common factor. By Equation (12), the dGAB copula serves to extend and generalize the two assumptions in Yang et al. (2009). To elaborate, we extend Assumptions A and B to accommodate scenarios encompassing two common factors, denoted as I and W, where individual random variables show conditional independence. Notably, the novel factor I assumes discrete values of either unity or zero with probabilities p and $1-p$, respectively. When $I = 1$, the individual random variables follow the Gaussian distribution with a distortion function. Conversely, when $I = 0$, the random variables follow the $C^{\mathcal{A},\mathcal{B}}$ copula, thereby signifying the potential emergence of tail risks. Furthermore, a discernible observation is that the bivariate marginal distributions manifest as 'distorted Gaussian-Fréchet' distributions, representing a mix distortion of Gaussian and $C^{\mathcal{A},\mathcal{B}}$ copula components. This intricate fusion renders the model more universally applicable and germane than the single $C^{\mathcal{A},\mathcal{B}}$ copula in terms of dependence modeling.

Let us define the function $F_j(u, i, v, \mathbf{x}) = \mathbb{P}(U_j \leq u, I = i, W \leq v, \mathbf{X} \leq \mathbf{x})$ to represents the joint distribution of U_j, W, I, and $\mathbf{x}$, where $(u, i, v, \mathbf{x}) \in [0,1] \times 0, 1 \times [0,1] \times \mathbb{R}^n$. Additionally, let $\mathbf{\Phi}_\Sigma$ denote the distribution of the multivariate normal random variables $\mathcal{N}(\mathbf{0}, \Sigma)$. Based on these notations, we present the following proposition.

Proposition 4.1. *For each random variable U_i defined by Equation (12), the joint distribution $F_j(u, i, v, \mathbf{x})$ is given by*

$$\begin{cases} (1-\alpha)\Big(a_{j1}M\big(E_j(u),v\big)+a_{j2}\Pi\big(E_j(u),v\big)+a_{j3}W\big(E_j(u),v\big)\Big)\mathbf{\Phi}_{I_n}(\mathbf{x}), & i=0, \\ \alpha v\mathbf{\Phi}_{\Sigma_j}\Big(\Phi^{-1}\big(D_j(u)\big),\mathbf{x}\Big), & i=1, \end{cases} \tag{13}$$

where I_n is the identity matrix of size n and $\Sigma_j = (\rho_{st})_{st}$ is a matrix of size $n+1$ satisfying $\rho_{st} = 1_{\{s=t\}} + a_{st}1_{\{s=1,t>1\}} + a_{ts}1_{\{s>1,t=1\}}$ for $s,t = 1, 2, \ldots, n+1$.

By Equation (13), the joint distribution of U_j, W, and $\{X_k\}_{k=1}^n$ can be regarded as a distorted "mixture" of the Gaussian copula and $C^{\mathcal{A},\mathcal{B}}$ copula. On the other hand, it also can be seen as an extension of the two assumptions in Yang et al. (2009) by noting that letting $i=0$, $\alpha=0$, $\mathbf{x}\to+\infty$, and $E_i(u)=u$ for $u\in[0,1]$ yields the joint distribution of U_j and W, i.e., $F_j(u,0,v,\ +\infty)=\ a_{j1}M(u,v)+a_{j2}\prod(u,v)+\ a_{j3}W(u,v)$.

Conversely, if the following two assumptions are satisfied, then we can prove that the joint distribution is exactly the dGAB copula.

Assumption G-A. *There exist random variables $W\sim \mathrm{U}(0,1), I\sim \mathrm{Bernoulli}(\alpha)$, and random vector $\{X_k\}_{k=1}^n \sim \mathcal{N}(0,I_n)$ such that random variables $U_1, U_2, \ldots, U_n$ are conditionally independent on them.*

Assumption G-B. *The joint distribution of U_k, I, W, and $\{X_k\}_{k=1}^n$ takes the distribution function (13).*

The theorem below characterizes the dGAB copula with the above two assumptions.

Theorem 4.3. *Suppose that Assumption G-A and Assumption G-B hold. Then the joint distribution of $(U_1, U_2, \ldots, U_n)$ is exactly the dGAB copula.*

It is important to emphasize that the probability structure is of critical importance in many topics of quantitative finance. Credit derivatives pricing often involves the dependence modeling between default times and the Gaussian copula is often selected for such purpose (Li, 2000). However, the Gaussian copula is not immune to criticism due to its inability to tail dependence. In this context, an alternative solution is found in the form of the dGAB copula, which retains the merits of the Gaussian copula while concurrently addressing the issue of tail dependence. We refer to the next section for the application to the BDS pricing.

5. Application and parameters estimation

In this section, we apply the dGAB copula to credit derivatives pricing and provide a parameter estimation method based on the classical EM algorithm.

5.1. The basket default swap pricing model

A basket default swap (BDS), also called m-th-to-default (NTD), is a credit default protection instrument written on a basket of n credits ($m\le n$). The protection buyer pays a periodic, pre-specified rate on a notional principal until the m-th-to-default credit in the basket occurs or the NTD contract expires. In return, if the m-th default occurs before the contract expiration, the buyer is entitled either to exchange the bond issued by the m-th defaulted entity for its face value N or to receive a cash equivalent payment given by $(1-R_m)N$, where R_m is called the recovery rate while $1-R_m$ is commonly referred to as loss given default for the m-th credit.

The fundamental theory for BDS pricing has been extensively discussed (Agosto & Ahelegbey, 2022; Arvanitis et al., 1999; Chen & Glasserman, 2008; Schönbucher, 2003). A critical aspect of the modeling process is the proper specification of the joint density of default times and their correlation or dependence. Copula factor models, such as the Gaussian copula factor model, offer advantages such as the potential of getting explicit or semi-explicit solutions, as demonstrated by Laurent and Gregory (2005) and Madan et al. (2006). However, the Gaussian copula has faced criticism for its lack of capability in capturing tail dependence, which can lead to significant tail risks undetected in the context of credit derivatives. In light of these limitations, we use the dGAB copula to model the dependence structure of default times in order to address the shortcomings of the Gaussian copula in this section.

For simplicity, we consider a homogeneous basket of assets, where all assets have a uniform recovery rate π and a notional amount of 1. Denote by τ_i the default time of the obligator $i=1,2,\ldots,n$, and denote by $\tau^{(i)}$ the order statistics of them, i.e. $\tau^{(1)}\le\tau^{(2)}\le\cdots\le\tau^{(n)}$. Let P represent the coupon rate, which is the periodic payment from the protection buyer to the protection seller, usually a percentage of the notional amount and paid at regular intervals like quarterly or annually. Moreover, we denote by $T>0$ the maturity, namely the duration of the contract. Denote by T_0, T_1, T_2, $\cdots$, and T_q the dates of coupon payments. With this setup, the default leg of an m-th to default swap can be evaluated as follows:

$$\begin{aligned}\text{Default Leg} &= \mathbb{E}\left[(1-\pi)1_{\{0\le\tau^{(m)}\le T\}}B\left(\tau^{(m)}\right)\right] \\ &= (1-\pi)\int_0^T B(t)\mathrm{d}F^{(m)}(t),\end{aligned} \tag{14}$$

where $B(t) = \exp(-rt)$ is the discount factor with r the risk-free interest rate, and $F^{(m)}$ is the distribution function of $\tau^{(m)}$. In particular, we note that

$$\begin{aligned}F^{(m)}(t) = \mathbb{P}\left(\tau^{(m)} \le t\right) &= \mathbb{P}(N(t) \ge m) \\ &= \sum_{k=m}^{n}\mathbb{P}(N(t) = k),\end{aligned}$$

where $N(t) = \sum_{i=1}^{n} 1_{\{\tau_i \le t\}}$ denotes the number of defaults by time t. On the other hand, the premium leg can be evaluated as

$$\begin{aligned}\text{Premium Leg} &= \mathbb{E}\left[P\sum_{j=1}^{q}B(T_{j-1})(T_j - T_{j-1})1_{\{N(T_j)<m\}}\right] \\ &= P\sum_{j=1}^{q}B(T_{j-1})(T_j - T_{j-1})\mathbb{P}(N(T_j) < m) \\ &= P\sum_{j=1}^{q}\sum_{k=0}^{m-1}B(T_{j-1})(T_j - T_{j-1})\mathbb{P}(N(T_j) = k).\end{aligned} \tag{15}$$

Therefore, the coupon rate P, which represents the price of the BDS, can be derived explicitly by equating the default leg (14) and the premium leg (15), i.e.

$$P = \frac{(1-\pi)\int_0^T B(t)\mathrm{d}F^{(m)}(t)}{\sum_{j=1}^{q}\sum_{k=0}^{m-1}B(T_{j-1})(T_j - T_{j-1})\mathbb{P}(N(T_j) = k)}. \tag{16}$$

The probability $\mathbb{P}(N(T) = k)$ is a crucial factor in BDS pricing. It can be calculated using Monte Carlo simulations due to the known probability structures of default times, as described in Example 1. In cases where the Gaussian copula takes on specific forms, such as the factor models, closed-form solutions are available, as detailed in the following section.

5.2. Factor model and closed-form solutions

As a special case of the Gaussian copula, the one-factor Gaussian copula has gained widespread recognition for its application in the valuation of basket default swaps (Dalla Valle et al., 2016; Laurent & Gregory, 2005; Moffatt, 2005). Specifically, within the context of the one-factor Gaussian copula model C_Σ, if the vector $(Z_1, Z_2, \ldots, Z_n)$ follows its distribution, it can be established that there exist independent standard Gaussian random variables, namely X_1, X_2, $\cdots$, X_n, and X, such that for $i = 1, 2, \ldots, n$, the relationship holds: $Z_i = \Phi\left(\zeta_i X_i + \sqrt{1-\zeta_i^2}X\right)$, where Φ is the cumulative distribution function of the standard Gaussian distribution and $\zeta_i \in (-1, 1)$ is a constant.

The $(W_1, W_2, \ldots, W_n)$ sequence defined by Equation (11), which follows the $C^{\mathcal{A},\mathcal{B}}$ copula, can be transformed into the $(Z_1, Z_2, \ldots, Z_n)$ sequence following the Gaussian copula through the application of Theorem 4.2. This results in the following random variables

$$T_i = I \cdot D_i^{-1}(Z_i) + (1-I) \cdot E_i^{-1}(W_i),$$

for $i = 1, 2, \ldots, n$, where $I \sim \text{Bernoulli}(\alpha)$ is independent of the rest random variables. In a BDS with n obligators, the default time is given by

$$\tau_i = F_i^{-1}\left(I \cdot D_i^{-1}(Z_i) + (1-I) \cdot E_i^{-1}(W_i)\right), \tag{17}$$

where F_i is the distribution function of default time τ_i for $i = 1, 2, \ldots, n$.

The dependence structure of the default times can be regarded as a three-factor model, as each default time τ_i is influenced by the common factors (X, I, W), as specified in Equation (17). This three-factor dependence structure facilitates the calculation of the conditional default probability $p_t^{i|X,I,W}$, which is defined as

$$p_t^{i|X,I,W} = \mathbb{P}(\tau_i \le t | X, I, W), \tag{18}$$

which represents the probability of the i-th obligor defaulting before time t, given values of the common factors (X, I, W).

Proposition 5.1. *If the default time τ_i is specified by Equation (17), the conditional default probability on (X, I, W) is given by*

$$p_t^{i|X,I,W} = \begin{cases} p_t^{i|X} = \Phi\left(\dfrac{\Phi^{-1}(D_i(F_i(t))) - \sqrt{1-\zeta_i^2}X}{\zeta_i}\right), & I = 1; \\ q_t^{i|W} = a_{i1}1_{\{W \le E_i(F_i(t))\}} + a_{i2}E_i(F_i(t)) + a_{i3}1_{\{W \ge 1 - E_i(F_i(t))\}}, & I = 0. \end{cases}$$

The three common factors (X, I, W) can be considered as latent variables that are not directly observable. As demonstrated by the proposition above, these three common factors also hold economic interpretations. Factor I follows a Bernoulli distribution and can only take on two values, namely 1 and 0, with probabilities α and $1-\alpha$, respectively. When $I = 1$, the conditional probability of default is described by normal distributions, signifying a "normal" state of the economic environment. Conversely, when $I = 0$, the conditional probability of default is modeled by a Fréchet copula (as stated in Theorem 2.1, Yang et al. (2009)), indicating the presence of potential tail risks. The other two factors, X and W, hold analogous interpretations corresponding to the two states $I = 1$ and $I = 0$. As indicated by Proposition 5.1, these factors may correspond to specific macroeconomic indicators under each state.

The proposition posits independence between $p_t^{i|X,1,W}$ and W, as well as between $p_t^{i|X,0,W}$ and X. We denote them as $p_t^{i|X}$ and $q_t^{i|W}$, respectively, for simplicity. Using these conditional probabilities, we proceed to derive the BDS price under the dGAB copula in the subsequent theorem.

Theorem 5.1 (BDS price under dGAB copula). *Suppose that the default time τ_i is specified by Equation (17). Then the BDS price can be given explicitly by*

$$P = \frac{(1-\pi)\sum_{k=m}^{n}\int_0^T B(t)\mathrm{d}\mathbb{P}(N(t)=k)}{\sum_{j=1}^{q}\sum_{k=0}^{m-1}B(T_{j-1})(T_j - T_{j-1})\mathbb{P}(N(T_j)=k)}, \tag{19}$$

where the probability $\mathbb{P}(N(t)=k)$ is defined as:

$$\mathbb{P}(N(t)=k) = \sum_{|I|=k}\left(\alpha\int_{-\infty}^{\infty}\prod_{i\in I}p_t^{i|x}\prod_{i\notin I}\left(1-p_t^{i|x}\right)\phi(x)\,\mathrm{d}x + (1-\alpha)\int_0^1\prod_{i\in I}q_t^{i|w}\prod_{i\notin I}\left(1-q_t^{i|w}\right)\mathrm{d}w\right).$$

In the above formulation, the set I denotes a subset of $\{1,2,\dots,n\}$, the notation $|I|$ denotes the cardinality of I, and functions $p_t^{i|x}$ and $q_t^{i|w}$ are defined in Proposition 5.1.

Remark. *According to Theorem 4.2, the probability structure of the dGAB copula has clear mathematical expressions. Specifically, if the Gaussian copula in the dGAB copula is reduced to a one-factor model (*Hull & White, 2004*), the probability structure of the dGAB copula forms a general two-factor model, with the first factor being derived from the one-factor model and the second factor arising from the $C^{\mathcal{A},\mathcal{B}}$ copula. This two-factor model can be applied to the valuation of BDS through the multi-factor model proposed by* Laurent and Gregory (2005)*.*

5.3. Numerical examples: Monte Carlo simulation vs closed-form solutions

In this section, we undertake a comparative analysis of the pricing impact stemming from the utilization of Monte Carlo simulation and closed-form solutions within the context of the dGAB copula. For numerical analysis, we take a continuous compounding rate denoted by $r = 5\%$ and consider a set of three obligators characterized by constant hazard rates[2] of 0.05, 0.01, and 0.02, respectively. We assume the maturity $T = 10$ and the dates of coupon payments as $T_i = (i-1)/10$ for $i = 1, 2, \dots, 100$. Concurrently, a recovery rate of 0.3 is applied for all obligators. In the initial phase of our investigation, we take a constant Spearman's rank correlation of 0.2 between default times. This model is similar to the illustrative examples found in the seminal work by Joshi and Kainth (2004) and Chen and Glasserman (2008).

We assume that the default times τ_1, τ_2, and τ_3 follow the dGAB copula. As indicated by Theorem 3.1, the dGAB copula can be determined by its bivariate dGF copulas of the following specific form:

$$\mathrm{dGF}_{ij}(u,v) = \alpha C_{\rho_{ij}}(D_2(u), D_2(v)) + (1-\alpha)C^{\mathrm{F}}_{d_{ij}^+, d_{ij}^-}(E_2(u), E_2(v)), \tag{20}$$

where $i \neq j \in \{1,2,3\}$ are two different indices, (D_2, E_2) is a pair of feasible distortion functions in Table 1, and $C^{\mathrm{F}}_{d_{ij}^+, d_{ij}^-}$ is the Fréchet copula defined by Equation (2). In the selection of distortion functions, we favor the pair (D_2, E_2) primarily for two reasons: (1) The distortion pair (D_2, E_2) encompasses only one parameter, denoted as α, thereby diminishing the computational costs involved in subsequent estimation processes. (2) The constraint imposed on α is comparatively straightforward in comparison to the constraints associated with the other two pairs detailed in Table 1, facilitating a more straightforward estimation procedure. Given that we have assumed a constant Spearman's rank correlation of 0.2 between default times, denoted as $\rho_{ij}^S = 0.2$ for all $i \neq j$, it follows that the correlation coefficient ρ_{ij} remains consistent across all pairs $i \neq j$. Additionally, for simplicity, we adopt the assumption that d_{ij}^+ and d_{ij}^- are the same for all $i \neq j$. Consequently, we can simplify our notation by removing the subscripts i and j from these parameters, representing them as (ρ, d^+, d^-) in our subsequent development.

Since the dGAB copula involves multiple parameters, i.e., $\vartheta = (\alpha, \rho, d^+, d^-)$, we need to introduce additional assumptions for the calibration procedure. Building upon the work of Li et al. (2014), we predefine two parameters, d^+ and d^-, in the Fréchet copula. Consequently, in the context of the dGF copula (20), the updated parameters become $\vartheta = (\alpha, \rho)$. The calibration process involves two primary steps. First, we establish the relationship between α and ρ using Spearman's rank correlation coefficient within the framework of the marginal copula (20). Second, guided by the principles outlined in Theorem 3.1, we compute the dGAB copula based on these marginal copulas for the given α.

With all parameters of the dGAB copula in place, we are well-prepared to calculate BDS prices

[2]The default time distribution is conventionally modeled as an exponential distribution, denoted as $F(t) = \mathbb{P}(\tau \le t) = 1 - \exp(-ht)$, where the constant h is the so-called hazard rate.

through two distinct methodologies: Monte Carlo simulation and a closed-form solution presented in Theorem 5.1. It is noteworthy that the probability structure outlined in Example 1 assumes a pivotal role in the BDS pricing process, elaborated upon in Algorithm 2 within Appendix J. Furthermore, alongside the Monte Carlo simulation, we offer a detailed pricing algorithm for the closed-form solution, presented in Algorithm 3 in the appendices.

The analysis was performed on a PC with 8GB of RAM and a 2.30 GHz CPU. Specifically, we employed Matlab, a computational software tailored for numerical analysis, supplemented by imported optimization and statistics toolboxes. The numerical results across various parameter configurations are presented in the following Table 3. Our analysis focuses on BDS pricing concerning the parameters α, d^+, and d^-. Notably, α represents the Gaussian copula's weight within the dGAB (or equivalently, GF) copula, while d^+ and d^- denote the weights of comonotonic and countermonotonic copulas in the Fréchet copula. This numerical investigation aims to elucidate the BDS price's relationship with these three parameters.

Notably, the table reveals several significant findings. First, the price under the single Gaussian copula consistently exhibits the lowest values when compared to the prices under the GAB and dGAB copulas. Second, under identical parameter values for α, d^+, and d^-, the price under the dGAB copula consistently surpasses that under the GAB copula. Third, when d^+ exceeds d^-, the corresponding BDS price experiences a notable increase. These findings collectively suggest that tail dependence exerts a critical influence on the BDS price, with greater tail dependence resulting in higher BDS prices. Finally, it is worth noting that the computational time associated with Algorithm 3 for the closed-form solutions is notably shorter than that required for the Monte Carlo simulation, thereby underscoring the advantages of employing the dGAB copula in BDS pricing.

5.4. Parameter estimation of dGAB copula

5.4.1. Dataset description

For the empirical study, we employ the daily log returns of two financial assets, namely, crude oil and the stock of American Airlines Group Inc. (AAL), spanning the period from January 1, 2006, to January 1, 2010. The rationale behind this selection is primarily grounded in the negative correlation observed between these two assets, which has the potential to manifest noteworthy sub-tail dependencies. The dataset, comprising 975 observations, was sourced from Yahoo Finance. Descriptive statistics and a scatter plot between the two log return series are presented in Table 4 and the left panel of Figure 2, respectively. Notably, the mean returns of the two indices are close to zero, while the returns exhibit kurtosis values greater than 3. In other words, the scatter plot proves to be insufficient in terms of providing meaningful insights.

The primary goal of this empirical investigation is to apply the dGAB (GAB) copula to fit the daily log returns. To construct the marginal models, we employ empirical distributions, which involve transforming the daily log returns $\mathbf{R}_t = \left(R_t^{(1)}, R_t^{(2)}\right)$ to pseudo samples $\mathbf{U}_t = \left(U_t^{(1)}, U_t^{(2)}\right)$, respectively. To be specific, we use the rank of $R_t^{(1)}$ and $R_t^{(2)}$ to calculate $U_t^{(1)}$ and $U_t^{(2)}$, which are then normalized by the dataset's length $n = 975$, namely

$$U_t^{(i)} = \frac{\text{rank}\left(R_t^i\right)}{n+1}, \text{for } i = 1, 2, t = 1, 2, \dots, n.$$

Additional details can be found in Chen and Fan (2006). The scatter plot of the pseudo samples is presented in the right panel of Figure 2.

The approximation model

Given that the dGAB (GAB) copula with $\Sigma \neq I_n$ can be uniquely identified by its bivariate marginals, as shown by Theorem 3.1, it is possible to utilize the bivariate marginal copulas to determine the copula.

Table 3. Comparison of Second-to-Default prices: Monte Carlo simulation (in blue) vs closed-form solution (in pink). The values in parentheses and "CT" denote the standard deviations and average computational time, respectively, derived from 5000 independent Monte Carlo simulations.

	α	0.3	0.5	0.7	0.9	CT
$(d^+, d^-) = (0.5, 0.5)$	P_{GAB}	1.18%	1.12%	1.08%	1.04%	1.93s
		(0.042%)	(0.041%)	(0.038%)	(0.040%)	
		1.19%	1.13%	1.10%	1.02%	0.03s
	P_{dGAB}	1.19%	1.13%	1.32%	1.47%	2.13s
		(0.043%)	(0.041%)	(0.044%)	(0.046%)	
		1.23%	1.12%	1.22%	1.41%	0.04s
$(d^+, d^-) = (0.8, 0.2)$	P_{GAB}	1.30%	1.19%	1.18%	1.05%	1.84s
		(0.043%)	(0.045%)	(0.045%)	(0.036%)	
		1.44%	1.23%	1.21%	1.11%	0.03s
	P_{dGAB}	1.26%	1.19%	1.39%	1.42%	2.12s
		(0.043%)	(0.045%)	(0.045%)	(0.036%)	
		1.45%	1.26%	1.29%	1.44%	0.04s
$P_{\text{Gaussian}} = 0.96\%$						0.02s

Table 4. Summary statistics of the daily log returns of crude oil and AAL from 2006 to 2010.

	Min	Median	Max	Mean	S.D.	Skewness	Kurtosis
Crude oil	−13.07%	0.13%	16.31%	0.02%	2.92%	0.16	7.20
AAL	−36.18%	−0.20%	46.21%	−0.20%	6.35%	0.20	8.39

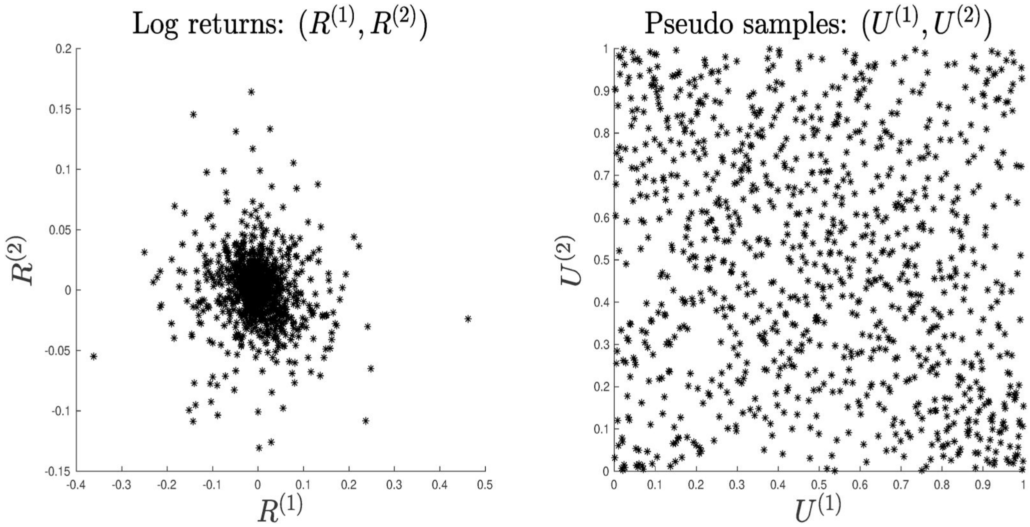

Figure 2. The scatter plot of the daily log returns of crude oil versus AAL from 2006 to 2010, and the right panel displays the scatter plot of the pseudo samples of the same dataset.

In this empirical study, we choose the first distortion function in Table 1, namely $D(u) = u^3$ as the distortion function in the Gaussian copula. Accordingly, we have the feasible pair $E(u) = (u - \alpha^{(1)}u^3)/(1-\alpha^{(1)})$ for $u \in [0,1]$, where $\alpha^{(1)}$ denotes the weight attributed to the Gaussian component copula within the dGAB copula. Therefore, the bivariate marginal copulas of the dGAB copula can be expressed as follows:

$$C(\mathbf{u}) = \alpha^{(1)}C_\rho(\mathbf{D}(\mathbf{u})) + \alpha^{(2)}M(\mathbf{E}(\mathbf{u})) + \alpha^{(3)}\Pi(\mathbf{E}(\mathbf{u})) + \alpha^{(4)}W(\mathbf{E}(\mathbf{u})), \tag{21}$$

where $\mathbf{u} = (u^{(1)}, u^{(2)}), \mathbf{D}(\mathbf{u}) = (D(u^{(1)}), D(u^{(2)}))$, $\mathbf{E}(\mathbf{u}) = (E(u^{(1)}), E(u^{(2)}))$, and $\alpha^{(i)}, i = 1,2,3,4,$ are non-negative satisfying $\sum_{i=1}^4 \alpha^{(i)} = 1$. Additionally, the correlation coefficient $\rho \in (-1, 1)$ is non-zero.

However, the comonotonic and countermonotonic copulas are singular copulas that do not possess density functions (Halmos, 2013; Nelsen, 2006). This singular property presents a challenge in estimating their parameters. As noted in the introduction, the comonotonic and countermonotonic copulas can be viewed as extreme cases of the bivariate Gaussian copula C_ρ when ρ approaches 1 and -1, respectively. To facilitate the estimation of the parameters, we opt for the use of Gaussian copulas with correlation coefficients close to 1 and -1, namely $\rho_- = -0.99$ and $\rho_+ = 0.99$, to substitute the countermonotonic and comonotonic copulas in Equation (21), respectively. Namely, we estimate paratermers $\alpha^{(i)}$ for $i = 1,2,3,4$ and ρ in the following approximated copula:

$$C_{\text{apx}}(\mathbf{u}) = \alpha^{(1)}C_\rho(\mathbf{D}(\mathbf{u})) + \alpha^{(2)}C_{\rho_+}(\mathbf{E}(\mathbf{u})) + \alpha^{(3)}\Pi(\mathbf{E}(\mathbf{u})) + \alpha^{(4)}C_{\rho_-}(\mathbf{E}(\mathbf{u})). \tag{22}$$

It is important to observe that in cases where distortion functions are simple, the copula defined in Equation (22) shares similarities with the Gaussian mixture model (GMM). However, there exist distinct differences between them. (1) GMM relies on Gaussian probability distributions for its constituent components, whereas the model described in Equation (22) utilizes copulas to represent the underlying distributions. (2) A notable enhancement in the copula defined by Equation (22) compared to GMM is its incorporation of distortion functions. This inclusion substantially enhances the model's flexibility and adaptability.

In light of the fact that this estimation process relies on an approximation, we provide a brief analysis of the associated errors in Appendix N.

dGAB-EM algorithm

The estimation of its parameters can be achieved through the classical expectation maximization (EM) algorithm. Specifically, let $\mathbf{u}_t = (U_t^{(1)}, U_t^{(2)})$ be the sample following the Gaussian copula in each group, i.e., $\mathbf{u} \sim C_\rho$. Denote by z the random variable following the categorical distribution, i.e., $z \sim \text{Categorical}(k, \boldsymbol{\alpha})$, where $k = 4$, $\boldsymbol{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)})$, $\alpha^{(i)} \geq 0$, and $\sum_{i=1}^4 \alpha^{(i)} = 1$.

For notation convenience, we denote $\boldsymbol{\theta} = (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, \rho), (\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}) = (\rho, \rho_+,$

$0, \rho_-)$, and let $(D_1, D_2, D_3, D_4) = (D, E, E, E)$ in the following developments. A maximum likelihood estimation can be applied:

$$\begin{aligned} l(\boldsymbol{\theta}) &= \sum_{t=1}^{n} \log p(\mathbf{u}_t; \boldsymbol{\theta}) \\ &= \sum_{t=1}^{n} \sum_{j=1}^{4} \log \left(p(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}) \cdot p(z_t = j; \boldsymbol{\theta}) \right), \end{aligned} \tag{23}$$

where $p(z_t = j; \boldsymbol{\theta}) = \alpha^{(j)}$ and $p(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}) = c_{\rho^{(j)}}(\mathbf{D}_j(\mathbf{u}_t)) = c_{\rho^{(j)}}\left(D_j\left(u_t^{(1)}\right), D_j\left(u_t^{(2)}\right)\right)$. Here c_ρ is the density function of the Gaussian copula C_ρ for $\rho \in (-1, 1)$, i.e.,

$$\begin{aligned} &c_\rho(u, v) \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \\ &\exp\left(-\frac{\rho^2\left(\Phi^{-2}(u) + \Phi^{-2}(v)\right) - 2\rho\Phi^{-1}(u)\Phi^{-1}(v)}{2(1-\rho^2)} \right) \\ &\frac{1}{\phi\left(\Phi^{-1}(u)\right)\phi\left(\Phi^{-1}(v)\right)}, \end{aligned}$$

where ϕ is the standard normal density function.

This study details the dGAB-EM algorithm, which is utilized for parameter estimation and is presented below.

Algorithm 1 dGAB-EM algorithm

Initializing parameters: $iter \leftarrow 0, \alpha_{iter}^{(i)} \leftarrow 0.25$, for $i = 1, 2, 3, 4$, and $\rho_{iter} \leftarrow 0$
while $iter = 0$ or $||\boldsymbol{\theta}_{iter} - \boldsymbol{\theta}_{iter-1}|| > \epsilon$ **do**
  dGAB-E step: $w_j^t = p\left(z_t = j | \mathbf{u}_t; \boldsymbol{\theta}_{iter}\right)$
  $iter \leftarrow iter + 1$
  dGAB-M step: $\boldsymbol{\theta}_{iter} = \operatorname{argmax}_{\boldsymbol{\theta}} \sum_{t=1}^{n} \sum_{j=1}^{4} w_j^t \log\left(p(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}) \cdot p(z_t = j; \boldsymbol{\theta}) / w_j^t\right)$
end while
return $\boldsymbol{\theta}_{iter}$

For more details about the algorithm, we refer to Appendix M.

Note that for the conditional probability in the dGAB-E step in the dGAB-EM algorithm, Bayes' theorem indicates that

$$\begin{aligned} p(z_t = j | \mathbf{u}_t; \boldsymbol{\theta}_{iter}) &= \frac{p\left(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}_{iter}\right) \cdot p(z_t = j; \boldsymbol{\theta}_{iter})}{\sum_{j=1}^{4} p\left(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}_{iter}\right) \cdot p(z_t = j; \boldsymbol{\theta}_{iter})} \\ &= \frac{c_{\rho_{iter}^{(j)}}(\mathbf{D}_j(\mathbf{u}_t)) \cdot \alpha_{iter}^{(j)}}{\sum_{j=1}^{4} c_{\rho_{iter}^{(j)}}(\mathbf{D}_j(\mathbf{u}_t)) \cdot \alpha_{iter}^{(j)}}. \end{aligned}$$

The results

Exclusively applying the Gaussian copula to model pseudo samples $\left(U_t^{(1)}, U_t^{(2)}\right)$ yielded an estimated correlation coefficient $\hat{\rho}_{\text{Gaussian}} = -0.17$. To conduct a more thorough parameter analysis, we set an error threshold of $\epsilon = 10^{-5}$ and utilized Algorithm 1 within the framework defined by Equation (22). The analysis was performed on a PC with 8GB of RAM and a 2.30 GHz CPU. The results are summarized in Table 5.

The findings disclosed in this analysis yield several noteworthy observations. To begin with, it is evident that the pseudo samples demonstrate a discernible negative correlation, as evidenced by the negative values of both $\hat{\rho}_{\text{Gaussian}}$ and $\hat{\rho}$. Second, it is essential to underscore that the Gaussian copula carries a relatively diminished weight, constituting less than 30% of the total correlation structure. This observation suggests that the reliance on the Gaussian copula as the sole model for capturing correlation may not be a suitable approach. Third, it is noteworthy that both coefficient estimates, $\widehat{\alpha^{(2)}}$ and $\widehat{\alpha^{(4)}}$, exhibit positive values, implying the presence of comonotonicity and countermonotonicity within the datasets. However, it is imperative to acknowledge that in the cases of GAB and dGAB, $\widehat{\alpha^{(4)}}$ surpasses $\widehat{\alpha^{(2)}}$, signifying a greater prevalence of countermonotonicity relative to comonotonicity in the dataset. Fourth, the estimation results manifest a conspicuous dissimilarity between ρ_{Gab} and ρ_{Gaussian}, underscoring the efficacy of modeling the central and tail segments separately to enhance both

Table 6. The coefficients of Fréchet copula in GAB copula (E.2).

Copula function	Coefficient
Comonotonic copula	$\tilde{d}_{ij}^{+}/(1-x)$
Independent copula	$(\tilde{d}_{ij}^{\perp} - x)/(1-x)$
Countermonotonic copula	$\tilde{d}_{ij}^{-}/(1-x)$

Table 7. Parameters estimation with different pairs of (ρ_+, ρ_-).

(ρ_+, ρ_-)	$\widehat{\alpha^{(1)}}$	$\widehat{\alpha^{(2)}}$	$\widehat{\alpha^{(3)}}$	$\widehat{\alpha^{(4)}}$	$\hat{\rho}$
$(1-10^{-1}, -1+10^{-1})$	0.2316	0.0994	0.4616	0.2074	−0.7215
$(1-10^{-2}, -1+10^{-2})$	0.2559	0.1043	0.4176	0.2222	−0.7211
$(1-10^{-3}, -1+10^{-3})$	0.2579	0.1027	0.4135	0.2259	−0.7211
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(1-10^{-7}, -1+10^{-7})$	0.2578	0.1026	0.4144	0.2252	−0.7211
$(1-10^{-8}, -1+10^{-8})$	0.2578	0.1025	0.4142	0.2255	−0.7211
$(1-10^{-9}, -1+10^{-9})$	0.2878	0.1025	0.4140	0.2257	−0.7211

Table 5. Comparisons of three competing models. 'LL', 'AIC', and 'CT' denote the log-likelihood, Akaike information criterion, and computational time, respectively.

Model	Estimation results	LL	AIC	CT
Gaussian	$\hat{\rho}_{\text{Gaussian}} = -0.17$	−452.99	907.98	0.01s
GAB	$\hat{\boldsymbol{\theta}} = (0.29, 0.06, 0.53, 0.12, -0.78)$	−352.60	713.20	17.96s
dGAB	$\hat{\boldsymbol{\theta}} = (0.26, 0.10, 0.42, 0.22, -0.72)$	−322.79	653.58	19.83s

flexibility and accuracy. Finally, the values of both LL and AIC in the table consistently indicate the ordering of dGAB > GAB > Gaussian in terms of the quality of fit, thereby affirming the superior performance of the dGAB (GAB) copula models over the singular Gaussian copula model.

6. Conclusions

This study presents an extension of the single Gaussian copula to the dGAB (GAB) copula, with the aim of improving its ability to describe tail dependence. The newly developed families of copulas retain the advantageous properties of the classic Gaussian copula, whereby they can be uniquely determined by their bivariate copulas, provided that appropriate conditions are met. Furthermore, the study provides an analysis of the probability structure of the distorted GAB copula. As a practical application, the dGAB copula model is employed in the BDS pricing. A parameter estimation method based on the EM algorithm is also proposed (Table 7). The results of our work suggest that the dGAB copula is a promising tool for modeling financial data with complex dependence structures.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix
Online Appendices

A. Some preliminaries

For readers' convenience, it is imperative that we incorporate two auxiliary lemmas from the existing literature.

Lemma A.1 (Yang et al. (2006)). *Each bivariate copula can be uniquely decomposed as a convex combination of a Fréchet copula and an indecomposable copula, i.e. for any bivariate copula C, we have the following decomposition:*

$$C(u,v) = \xi_1 M(u,v) + \xi_2 uv + \xi_3 W(u,v) + lf(u,v), \tag{A.1}$$

where the coefficients ξ_1, ξ_2, ξ_3, and l are uniquely determined, and f is an indecomposable copula.

Additionally, explicit formulas can be utilized to calculate the coefficients ξ_1, ξ_2, ξ_3, and l. For instance, Yang et al. (2006) indicates that the coefficient ξ_2 can be obtained through the expression $\xi_2 = \inf_{(u,v)\in[0,1]^2} \partial^2 C(u,v)/\partial u \partial v$. It is noteworthy that the bivariate Gaussian copula, characterized by a correlation coefficient $\rho \neq 0$, is indecomposable. Specifically, in the case where $\sigma \neq 0$, the coefficients ξ_1, ξ_2, and ξ_3 equal zero, while l equals one.

The second lemma pertains to the decomposition of a copula.

Lemma A.2 (Halmos, 2013; Nelsen, 2006). *For any bivariate copula C, it can be decomposed into a sum of an absolutely continuous item A and a singular item S uniquely, i.e.*

$$C(u,v) = A(u,v) + S(u,v),$$

where

$$A(u,v) = \int_0^u \int_0^v \frac{\partial^2 C(s,t)}{\partial s \partial t}\, \mathrm{d}s\mathrm{d}t \text{ and } S(u,v) = C(u,v) - A(u,v).$$

B. Proof of Theorem 3.1

Proof. For the given dGAB copula of the following form:

$$\mathrm{dGAB}(\mathbf{u}) = \alpha C_\Sigma(\mathbf{D}(\mathbf{u})) + (1-\alpha) C^{\mathcal{A},\mathcal{B}}(\mathbf{E}(\mathbf{u})), \mathbf{u} \in [0,1]^n,$$

the objective of this poof is to determine $\Sigma = (\rho_{ij})$, α, a_{ij}, D_i, and E_i for $i = 1,2,\dots,n$ and $j = 1, 2, 3$ from its bivariate marginal copulas. It is obvious that for different indices i and j, the bivariate marginal copula takes the following form:

$$\mathrm{dGAB}_{ij}(u,v) = \eta_1 C_{\rho_{ij}}(D_i(u), D_j(v)) + \eta_2 M(E_i(u), E_j(v)) + \eta_3 E_i(u) E_j(v) + \eta_4 W(E_i(u), E_j(v)),$$

where $\eta_1 = \alpha, \eta_2 = (1-\alpha)(a_{i1}a_{j1} + a_{i3}a_{j3}), \eta_4 = (1-\alpha)(a_{i1}\ a_{j3} + a_{i3}a_{j1})$, and $\eta_3 = 1 - (\eta_1 + \eta_2 + \eta_4)$.

Therefore, it is equivalent to determining the coefficients η_i for $i = 1,2,3,4$ and ρ_{ij} through the bivariate marginal copula dGAB_{ij}.

Applying Lemma A.1 to the marginal copula dGAB_{ij} yields the following decomposition:

$$\mathrm{dGAB}_{ij}(u,v) = lf_{ij}(u,v) + \xi_1 M(u,v) + \xi_2 uv + \xi_3 W(u,v), \tag{B.1}$$

where the coefficients l, ξ_1, ξ_2, ξ_3, and the indecomposable copula $f_{ij}(u,v)$ can be uniquely determined.

For the copula f_{ij}, it can be decomposed into a sum of an absolutely continuous part and a singular part by Lemma A.2, namely $f_{ij}(u,v) = A_{ij}(u,v) + S_{ij}(u,v)$, where $A(u, v)$ and $S(u, v)$ denote the absolutely continuous and singular parts, respectively. Substitute the decomposition into Equation (B.1) gives

$$\begin{aligned}&\eta_1 C_{\rho_{ij}}(D_i(u), D_j(v)) + \eta_2 M(E_i(u), E_j(v)) + \eta_3 E_i(u)E_j(v) + \eta_4 W(E_i(u), E_j(v)) \\ &= lA_{ij}(u,v) + lS_{ij}(u,v) + \xi_1 M(u,v) + \xi_2 uv + \xi_3 W(u,v).\end{aligned} \tag{B.2}$$

Note that absolutely continuous items in Equation (B.2) are $C_{\rho_{ij}}(D_i(u), D_j(v)), E_i(u)E_j(v)$, $A(u, v)$, and uv, while singular items are $M(E_i(u), E_j(v)), W(E_i(u), E_j(v))$, $S(u, v)$, $M(u, v)$, and $W(u, v)$. By Lemma A.2, we know that the decomposition of a copula into an absolutely continuous part and a singular part is unique. Therefore, we have the following two equations:

$$M(E_i(u), E_j(v)), W(E_i(u), E_j(v)) \tag{B.3}$$

$$\eta_1 C_{\rho_{ij}}(D_i(u), D_j(v)) + \eta_3 E_i(u)E_j(v) = \xi_2 uv + lA_{ij}(u,v), \tag{B.4}$$

for all $(u,v) \in [0,1]^2$.

Letting $v = 1$ in Equation (B.3) yields that $(\eta_2 + \eta_4)E_i(u) = (\xi_1 + \xi_3)u + lS_{ij}(u,1)$ for all $u \in [0,1]$. Furthermore, letting $u = 1$ yields

$$\eta_2 + \eta_4 = \xi_1 + \xi_3 + lS_{ij}(1,1). \tag{B.5}$$

Accordingly, for all $u \in [0,1]$, we have

$$E_i(u) = \frac{(\xi_1 + \xi_3)u + lS_{ij}(u,1)}{\xi_1 + \xi_3 + lS_{ij}(1,1)},$$

thus the distortion function E_i is uniquely determined for $i = 1,2,\dots,n$.

In the following, we will proceed to determine the coefficients η_2 and η_4. Since distortion functions are always continuous, there exists a pair of positive numbers $(u_0,\ v_0)$ which are sufficiently small such that $W(E_i(u_0), E_j(v_0)) = 0$ and $W(u_0, v_0) = 0$. Substituting such a pair $(u_0,\ v_0)$ into Equation (B.3), we have $\eta_2 M(E_i(u_0), E_j(v_0)) = \xi_1 M(u_0, v_0) + lS_{ij}(u_0, v_0)$, then we have

$$\eta_2 = \frac{\xi_1 M(u_0,v_0) + lS_{ij}(u_0,v_0)}{M(E_i(u_0), E_j(v_0))}.$$

Accordingly, the coefficient η_4 can also be determined by Equation (B.5), i.e.

$$\eta_4 = \xi_1 + \xi_3 + lS_{ij}(1,1) - \frac{\xi_1 M(u_0,v_0) + lS_{ij}(u_0,v_0)}{M(E_i(u_0), E_j(v_0))}.$$

To summarize, the coefficients η_2, η_4, and distortion function E_i have been uniquely determined now. Next, we proceed to determine the remaining coefficients η_1, η_3, and distortion functions D_i for $i = 1,2,\dots,n$.

By letting $u = v = 1$ in Equation (B.4), we obtain that

$$\eta_1 + \eta_3 = \xi_2 + lA_{ij}(1,1). \tag{B.6}$$

On the other hand, a rearrangement of the relationship $\eta_1 D_i(u) + (1-\eta_1)E_i(u) = u$ yields

$$D_i(u) = \frac{u-(1-\eta_1)E_i(u)}{\eta_1}, \text{for all } u \in [0,1]. \tag{B.7}$$

Substituting (u, v) with $\left(E_i^{-1}(u), E_j^{-1}(v)\right)$ in Equation (B.4) yields that for all $(u,v) \in [0,1]^2$,

$$\eta_1 C_{\rho_{ij}}\left(D_i\left(E_i^{-1}(u)\right), D_j\left(E_j^{-1}(v)\right)\right) + \eta_3 uv = \xi_2 E_i^{-1}(u)E_j^{-1}(v) + lA_{ij}\left(E_i^{-1}(u), E_j^{-1}(v)\right). \tag{B.8}$$

Taking the second mixed partial derivative for both sides in Equation (B.8) yields

$$\eta_1 \frac{\partial^2 C_{\rho_{ij}}\left(D_i\left(E_i^{-1}(u)\right), D_j\left(E_j^{-1}(v)\right)\right)}{\partial u \partial v} + \eta_3 = \xi_2 \frac{\mathrm{d}E_i^{-1}(u)}{\mathrm{d}u}\frac{\mathrm{d}E_j^{-1}(u)}{\mathrm{d}u} + l\frac{\partial^2 A_{ij}\left(E_i^{-1}(u), E_j^{-1}(v)\right)}{\partial u \partial v}, \tag{B.9}$$

for all $(u,v) \in [0,1]^2$.

By direct calculation we have

$$\frac{\partial^2 C_{\rho_{ij}}\left(D_i\left(E_i^{-1}(u)\right), D_j\left(E_j^{-1}(v)\right)\right)}{\partial u \partial v} = \frac{\mathrm{d}D_i\left(E_i^{-1}(u)\right)}{\mathrm{d}u}\frac{\mathrm{d}D_j\left(E_j^{-1}(v)\right)}{\mathrm{d}v} c_{\rho_{ij}}\left(D_i\left(E_i^{-1}(u)\right), D_j\left(E_j^{-1}(v)\right)\right), \tag{B.10}$$

where c_ρ is the density function of the Gaussian copula, i.e.

$$c_\rho(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\Phi^{-2}(u)+\Phi^{-2}(v)-2\rho\Phi^{-1}(u)\Phi^{-1}(v)}{2(1-\rho^2)}\right) \frac{1}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}.$$

Since we assume that $\Sigma \neq I_n$, there exists at least a non-zero correlation coefficient ρ_{ij} for $i \neq j$. In the following, we discuss two cases: $\rho_{ij} \neq 0$ and $\rho_{ij} = 0$ for $i \neq j$.

1. For the case $\rho_{ij} \neq 0$, we note that $\lim_{u\to 1^-, v\to 0^+} c_{\rho_{ij}}\left(D_i\left(E_i^{-1}(u)\right), D_j\left(E_j^{-1}(v)\right)\right) = 0$ holds. On the other hand, noting that $\mathrm{d}D_i\left(E_i^{-1}(u)\right)/\mathrm{d}u > 0$ and $\mathrm{d}D_j\left(E_j^{-1}(v)\right)/\mathrm{d}v > 0$ for all $u,v \in [0,1]$, Equation (B.10) indicates that

$$\inf_{(u,v)\in[0,1]^2} \frac{\partial^2}{\partial u \partial v} C_{\rho_{ij}}\left(D_i\left(E_i^{-1}(u)\right), D_j\left(E_j^{-1}(v)\right)\right) = 0.$$

Therefore, taking infimum on both sides of Equation (B.9) yields

$$\eta_3 = \inf_{(u,v)\in[0,1]^2}\left(\xi_2 \frac{\mathrm{d}E_i^{-1}(u)}{\mathrm{d}u}\frac{\mathrm{d}E_j^{-1}(v)}{\mathrm{d}v} + l\frac{\partial^2 A_{ij}\left(E_i^{-1}(u), E_j^{-1}(v)\right)}{\partial u \partial v}\right), \tag{B.11}$$

which shows that the coefficient η_3 has been uniquely determined for the case $\rho_{ij} \neq 0$, and thus, another coefficient η_1 can be determined by Equation (B.6), i.e. $\eta_1 = \xi_2 + lA_{ij}(1,1) - \eta_3$, which is exactly the weight α in the dGAB copula. Accordingly, the distortion function D_i can be determined by Equation (B.7) for $i = 1, 2, \ldots, n$.

2. For the case $\rho_{ij} = 0$, we note that Equation (B.4) indicates that $\eta_1 D_i(u)D_j(v) + \eta_3 E_i(u)E_j(v) = \xi_2 uv + lA_{ij}(u,v)$ holds for all $u, v \in [0,1]$. Since $\eta_1 = \alpha$, D_i, and E_i have already been determined previously, we can obtain the value of η_3 easily by choosing a pair of $(u_0, v_0) \in [0,1]^2$ in Equation (B.4).

Next, we proceed to determine the correlation coefficient ρ_{ij} in the bivariate marginal copula. A rearrangement of Equation (B.4) gives the Gaussian copula with correlation coefficient ρ_{ij} explicitly as

$$C_{\rho_{ij}}(u,v) = \frac{\xi_2 D_i^{-1}(u)D_j^{-1}(v) + lA_{ij}\left(D_i^{-1}(u), D_j^{-1}(v)\right) - \eta_3 E_i\left(D_i^{-1}(u)\right)E_j\left(D_j^{-1}(u)\right)}{\eta_1}. \tag{B.12}$$

On the other hand, the definition of the Gaussian indicates that

$$\frac{\partial C_{\rho_{ij}}(u,v)}{\partial u} = \Phi\left(\frac{\Phi^{-1}(v) - \rho_{ij}\Phi^{-1}(u)}{\sqrt{1-\rho_{ij}^2}}\right), \text{for all } u, v \in [0,1]^2. \tag{B.13}$$

By Equation (B.12), the left-hand side of Equation (B.13) is already known. Therefore, by choosing two different pairs (u_1, v_1) and (u_2, v_2) and substituting them into Equation (B.13), the correlation coefficient ρ_{ij} can be obtained by solving the system of equations.

To conclude, the distortion functions D_i and E_i, coefficients η_i, and correlation matrix Σ have all been uniquely determined from the bivariate marginal copulas now. Furthermore, the coefficients a_{ij} can be determined by η_1, η_2, and η_4 by Yang et al. (2009). The proof is completed. □

C. *Proof of Theorem 3.2*

Proof. Similar to the previous proof, the objective of this poof is to determine α, a_{ij}, D_i, and E_i for $i = 1, 2, \ldots, n$ and $j = 1, 2, 3$ in the dGAB copula from its bivariate marginal copulas dGAB_{ij} for $i \neq j$, which can be expressed as

$$\mathrm{dGAB}_{ij}(u,v) = \eta_1 D_i(u_i)D_j(u_j) + \eta_2 M\left(E_i(u_i), E_j(u_j)\right) + \eta_3 E_i(u_i)E_j(u_j) + \eta_4 W\left(E_i(u_i), E_j(u_j)\right),$$

where $\eta_1 = \alpha, \eta_2 = (1-\alpha)(a_{i1}a_{j1} + a_{i3}a_{j3}), \eta_4 = (1-\alpha)(a_{i1}a_{j3} + a_{i3}a_{j1})$, and $\eta_3 = 1 - (\eta_1 + \eta_2 + \eta_4)$. Now it suffices to determine D_i, E_i for $i = 1, 2, \ldots, n$ and η_i for $i = 1, 2, 3, 4$.

Applying Lemma A.1 to the marginal copula dGAB_{ij} for $i \neq j$ yields the decomposition:

$$\mathrm{dGAB}_{ij}(u,v) = lf_{ij}(u,v) + \xi_1 M(u,v) + \xi_2 uv + \xi_3 W(u,v), \tag{C.1}$$

where the coefficients l, ξ_1, ξ_2, and ξ_3 can be uniquely determined.

By Lemma A.2, the indecomposable copula f_{ij} can be decomposed into a convex sum of an absolutely continuous part and a singular part uniquely, i.e. $f_{ij}(u,v) = A_{ij}(u,v) + S_{ij}(u,v)$, where $A(u,\ v)$ and $S(u,\ v)$ denote the absolutely continuous and singular parts, respectively. Substituting $f(u,\ v)$ with $A(u,\ v)$ and $S(u,\ v)$ into Equation (C.1), we obtain

$$\eta_1 D_i(u)D_j(v) + \eta_2 M\big(E_i(u),E_j(v)\big) + \eta_3 E_i(u)E_j(v) + \eta_4 W\big(E_i(u),E_j(v)\big) = lA_{ij}(u,v) + lS_{ij}(u,v) + \xi_1 M(u,v) + \xi_2 uv + \xi_3 W(u,v). \tag{C.2}$$

Note that the absolutely continuous items in Equation (C.2) are $D_i(u)D_j(v), E_i(u)E_j(v)$, $A(u,\ v)$, and uv, and the singular items are $M\big(E_i(u),E_j(v)\big), W\big(E_i(u),E_j(v)\big)$, $S(u,\ v)$, $M(u,\ v)$, and $W(u,\ v)$. Lemma A.2 indicates that the decomposition into the absolutely continuous and the singular items is unique. Thus for all $(u,v) \in [0,1]^2$, we have

$$\eta_2 M\big(E_i(u),E_j(v)\big) + \eta_4 W\big(E_i(u),E_j(v)\big) = \xi_1 M(u,v) + \xi_3 W(u,v) + lS_{ij}(u,v), \tag{C.3}$$

$$\eta_1 D_i(u)D_j(v) + \eta_3 E_i(u)E_j(v) = \xi_2 uv + lA_{ij}(u,v). \tag{C.4}$$

Letting $v = 1$ in Equation (C.3) yields $(\eta_2 + \eta_4)E_i(u) = (\xi_1 + \xi_3)u + lS_{ij}(u,1)$ for all $u \in [0,1]$, and letting $u = v = 1$ in Equation (C.3) yields that

$$\eta_2 + \eta_4 = \xi_1 + \xi_3 + lS_{ij}(1,1). \tag{C.5}$$

Then for all $u \in [0,1]$, we have

$$E_i(u) = \frac{(\alpha+\gamma)u + lS(u,1)}{\alpha+\gamma+lS(1,1)},$$

thus the distortion function E_i is uniquely determined for $i = 1,2,\ldots,n$.

In the following, we will proceed to determine η_2 and η_4, respectively. Let $(u_0,\ v_0)$ be a pair of positive numbers that are sufficiently small such that $W\big(E_i(u_0),E_j(v_0)\big) = 0$ and $W(u_0,v_0) = 0$. Substituting $(u_0,\ v_0)$ into Equation (C.3), we obtain that $\eta_2 M\big(E_i(u_0),E_j(v_0)\big) = \xi_1 M(u_0,v_0) + lS_{ij}(u_0,v_0)$. Thus the coefficient η_2 can be uniquely determined by

$$\eta_2 = \frac{\xi_1 M(u_0,v_0) + lS_{ij}(u_0,v_0)}{M\big(E_i(u_0),E_j(v_0)\big)}. \tag{C.6}$$

Accordingly, the coefficient η_4 can be determined by Equation (C.5), namely

$$\eta_4 = \xi_1 + \xi_3 + lS_{ij}(1,1) - \frac{\xi_1 M(u_0,v_0) + lS_{ij}(u_0,v_0)}{M\big(E_i(u_0),E_j(v_0)\big)}.$$

Up to now, the coefficients η_2 and η_4, and distortion functions E_i have been uniquely determined. We proceed to determine the remaining coefficients ξ_1 and ξ_3, and distortion functions D_i.

By letting $u = v = 1$ in Equation (C.4), we obtain that $\eta_1 + \eta_3 = \xi_2 + lA_{ij}(1,1)$. On the other hand, since $\eta_1 D_i(u) + (1-\eta_1)E_i(u) = u$ holds for all $u \in [0,1]$, we have $D_i(u) = \big(u - (1-\eta_1)E_i(u)\big)/\eta_1$. Substituting into Equation (C.4) yields

$$\frac{\big(u-(1-\eta_1)E_i(u)\big)\big(v-(1-\eta_1)E_j(v)\big)}{\eta_1} + \eta_3 E_i(u)E_j(v) = \xi_2 uv + lA_{ij}(u,v).$$

Solving the equation for η_1, we obtain that

$$\eta_1 = \frac{(E_i(u)-u)(v-E_j(v))}{E_i(u)v + E_j(v)u + \big(\xi_2 + lA_{ij}(1,1) - 2\big)E_i(u)E_j(v) - \xi_2 uv - lA_{ij}(u,v)}. \tag{C.7}$$

By the assumption that there at least exist two nontrivial distortion functions, there exist two indices $i \neq j$ and a pair of $(u_0,\ v_0)$ such that the denominator in Equation (C.7) does not equal zero, hence the coefficient η_1 can be obtained. Accordingly, we have $\eta_3 = \xi_2 + lA_{ij}(1,1) - \eta_1$. Accordingly, the distortion function D_i can be uniquely determined by $D_i(u) = (u - \eta_1 E_i(u))/\eta_1$ for all $u \in [0,1]$.

To conclude, the distortion functions D_i, E_i, and coefficients η_i have all been uniquely determined by the bivariate marginal copulas. Furthermore, the coefficients a_{ij} can be determined by η_1, η_2, and η_4 by Yang et al. (2009). The proof is completed. □

D. Proof of Theorem 3.3

Proof. We first prove the sufficiency. Suppose that there exist non-negative constants $\alpha \in (0,1)$ and a_{ij} satisfying Equation (6). For each $i < n$, let W and V_i be uniform $[0,1]$ random variables, $\mathbf{X} = (X_1,X_2,\ldots,X_n) \in \mathcal{N}(\mathbf{0},I_n)$ be a standard normal random vector, and $I \sim \text{Bernoulli}(\alpha)$ be a Bernoulli random variable. Moreover, let $\big(A_i^+,A_i^\perp,A_i^-\big)$ be a random partition of the probability space satisfying that $\mathbb{P}\big(A_i^+\big) = a_{i1}, \mathbb{P}\big(A_i^\perp\big) = a_{i2}$, and $\mathbb{P}(A_i^-) = a_{i3}$. Assume that W, V_i, $\big(A_i^+,A_i^-,A_i^\perp\big)$, X_i, and I for $i = 1,2,\ldots,n$ are independent. Define

$$U_i = I\cdot D_i^{-1}\left(\Phi\left(\sum_{j=1}^n a_{im}X_m\right)\right) + (1-I) \cdot E_i^{-1}\Big(W1_{A_i^+} + V_i 1_{A_i^\perp} + (1-W)1_{A_i^-}\Big).$$

By Theorem 4.2 and Example 1, the joint distribution of them is exactly the dGAB copula. On the other hand, we see that U_i, $i = 1,2,\ldots,n$ are conditionally independent on $(W,I,\mathbf{X})$, and the joint distribution of U_i, I, W, and $\mathbf{X}$ is characterized by Equation (13). By letting $\alpha_{im} = \alpha$ for all $i,m = 1,2,\ldots,n$, Theorem 4.3 indicates that $\mathbb{P}(U_i \le u, U_m \le v) = \text{dGF}_{im}(u,v)$, which completes the proof of the sufficiency.

For the necessity direction, suppose that there exist uniform random variables U_i, $i = 1,2,\ldots,n$, following the dGAB copula such that $\mathbb{P}(U_i \le u, U_m \le v) = \text{dGF}_{im}(u,v)$. By Example 1, there exist uniform $[0,1]$ random variables W_i, $i = 1,2,\ldots,n$, and a_{ij}, $i = 1,2,\ldots,n$, $j = 1,\ 2,\ 3$, such that $(W_1,W_2,\ldots,W_n)$ follows the dGAB copula. Accordingly, $(W_1,W_2,\ldots,W_n)$ and $(U_1,U_2,\ldots,U_n)$ have the same distribution, which yields $\mathbb{P}(U_i \le u, U_m \le v) = \mathbb{P}(W_i \le u, W_m \le v) = \text{dGF}_{im}(u,v)$, then Equation (6) holds true, which completes the proof of the necessity. □

E. Proof of Proposition 3.1

Proof. Since $\Sigma = (\rho_{ij}) = I_n$, we have $\rho_{ij} = 0$ for all $i \neq j$. The bivariate marginal copula GAB_{ij} of the GAB copula takes the following form:

$$\begin{aligned}\mathrm{GAB}_{ij}(u,v) &= \alpha\Pi(u,v) + (1-\alpha)\left(d_{ij}^{+}M(u,v) + d_{ij}^{\perp}\Pi(u,v) + d_{ij}^{-}W(u,v)\right)\\ &= (1-\alpha)d_{ij}^{+}M(u,v) + \left(\alpha + (1-\alpha)d_{ij}^{\perp}\right)\Pi(u,v) + (1-\alpha)d_{ij}^{-}W(u,v)\\ &= \tilde{d}_{ij}^{+}M(u,v) + \tilde{d}_{ij}^{\perp}\Pi(u,v) + \tilde{d}_{ij}^{-}W(u,v),\end{aligned} \tag{E.1}$$

where $\tilde{d}_{ij}^{+} = (1-\alpha)d_{ij}^{+}, \tilde{d}_{ij}^{\perp} = \alpha + (1-\alpha)d_{ij}^{\perp}$, and $\tilde{d}_{ij}^{-} = (1-\alpha)d_{ij}^{-}$. In particular, we note that $d_{ij}^{+}, d_{ij}^{\perp}$, and d_{ij}^{-} are coefficients from the $C^{\mathcal{A},\mathcal{B}}$ copula satisfying that $d_{ij}^{+} = a_{i1}a_{j1} + a_{i3}a_{j3}, d_{ij}^{-} = a_{i1}a_{j3} + a_{i3}a_{j1}$, and $d_{ij}^{\perp} = 1 - d_{ij}^{+} - d_{ij}^{-}$.

Notably, $\tilde{d}_{ij}^{\perp} > 0$ for all $i \neq j$ since it is the sum of the coefficient of the independent Gaussian copula and the $C^{\mathcal{A},\mathcal{B}}$ copula. Let $x \in \left(0, \min_{i\neq j}\tilde{d}_{ij}^{\perp}\right)$, then we can reformulate GAB_{ij} as

$$\begin{aligned}\mathrm{GAB}_{ij}(u,v) = xuv + (1-x)\Bigg(&\frac{\tilde{d}_{ij}^{+}}{1-x}M(u,v) + \frac{\tilde{d}_{ij}^{\perp}-x}{1-x}\Pi(u,v)\\ &+ \frac{\tilde{d}_{ij}^{-}}{1-x}W(u,v)\Bigg).\end{aligned} \tag{E.2}$$

Therefore, the coefficients of the Fréchet copula in (E.2) can be summarized in the following Table 6.

We claim that there exists $x_0 \in \left(0, \min_{i\neq j}\tilde{d}_{ij}^{\perp}\right)$ such that the coefficients in Table 6 satisfy the $C^{\mathcal{A},\mathcal{B}}$ decomposable condition (Yang et al., 2006). Indeed, there exist non-negative constants $\tilde{a}_{ij}$ for $i = 1, 2, ..., n$ and $j = 1$, 2, 3 satisfying $\sum_{j=1}^{3}\tilde{a}_{ij} = 1$ such that the following equations hold, i.e.,

$$\frac{\tilde{d}_{ij}^{+}}{1-x_0} = \tilde{a}_{i1}\tilde{a}_{j1} + \tilde{a}_{i3}\tilde{a}_{j3}, \tag{E.3}$$

$$\frac{\tilde{d}_{ij}^{-}}{1-x_0} = \tilde{a}_{i1}\tilde{a}_{j3} + \tilde{a}_{i3}\tilde{a}_{j1}, \tag{E.4}$$

$$\frac{\tilde{d}_{ij}^{\perp}-x_0}{1-x_0} = 1 - \frac{\tilde{d}_{ij}^{+}}{1-x_0} - \frac{\tilde{d}_{ij}^{-}}{1-x_0}. \tag{E.5}$$

Now the corresponding GAB copula with marginals C_{ij} can be expressed as

$$\begin{aligned}\mathrm{GAB}(\mathbf{u}) = x_0\prod_{i=1}^{n}u_i + (1-x_0)\sum_{j_1=1}^{3}\sum_{j_2=1}^{3}\cdots\sum_{j_n=1}^{3}\left(\prod_{i=1}^{n}a_{ij_i}\right)\\ W\Big(\min_{i\le n, j_i=1}\{u_i\}, \min_{i\le n, j_i=3}\{u_i\}\Big)\prod_{i\le n, j_i=2}u_i.\end{aligned}$$

It is noted that the value of x_0 plays a critical role in the above copula. Indeed, the above bivariate copula can be reformulated as

$$\mathrm{GAB}_{ij}(u,v) = x_1\Pi(u,v) + (1-x_1)\left(\frac{\tilde{d}_{ij}^{+}}{1-x_1}M(u,v) + \frac{\tilde{\beta}_{ij}-x_1}{1-x_1}\Pi(u,v) + \frac{\tilde{\gamma}_{ij}}{1-x_1}W(u,v)\right),$$

where $0 < x_1 < \min\{\min_{i\neq j}\{\tilde{d}_{ij}^{\perp}\}, x_0\}$.

On the other hand, by Equations (E.3), (E.4), and (E.5), we have

$$\begin{aligned}\frac{\tilde{d}_{ij}^{+}}{1-x_1} &= \left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{i1}\right)\left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{j1}\right) + \left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{i3}\right)\left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{j3}\right),\\ \frac{\tilde{d}_{ij}^{-}}{1-x_1} &= \left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{i1}\right)\left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{j3}\right) + \left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{i3}\right)\left(\sqrt{\frac{1-x_0}{1-x_1}}\tilde{a}_{j1}\right),\\ \frac{\tilde{d}_{ij}^{\perp}-x_1}{1-x_1} &= 1 - \frac{\tilde{d}_{ij}^{+}}{1-x_1} - \frac{\tilde{d}_{ij}^{-}}{1-x_1},\end{aligned}$$

which also satisfy the $C^{\mathcal{A},\mathcal{B}}$ decomposable condition. Therefore, we obtain another GAB copula as

$$\mathrm{GAB}^{(1)}(\mathbf{u}) = x_1\prod_{i=1}^{n}u_i + \sqrt{(1-x_0)(1-x_1)}\sum_{j_1=1}^{3}\sum_{j_2=1}^{3}\cdots\sum_{j_n=1}^{3}\left(\prod_{i=1}^{n}a_{ij_i}\right)W\Big(\min_{j_i=1}\{u_i\}, \min_{j_i=3}\{u_i\}\Big)\prod_{j_i=2}u_i.$$

Obviously, GAB is not identical to $\mathrm{GAB}^{(1)}$. Therefore, we can conclude that the bivariate marginal copula (E.1) cannot uniquely determine a joint $C^{\mathcal{A},\mathcal{B}}$ copula, which completes the proof. □

F. Proof of Theorem 4.1

Proof. By definition of the tail coefficients, the lower tail and upper dependence coefficients are given by

$$\begin{aligned}
\lambda_L(\mathrm{dGAB}_{ij}) &= \lim_{u\to0^+}\frac{\mathrm{dGAB}_{ij}(u,u)}{u} = \lim_{u\to0^+}\frac{\alpha C_\Sigma(D(u),D(u)) + (1-\alpha)C^{\mathcal{A},\mathcal{B}}(E(u),E(u))}{u}\\
&= \lim_{u\to0^+}\alpha\frac{C_\Sigma(D(u),D(u))}{D(u)}\frac{D(u)}{u} + (1-\alpha)\frac{C^{\mathcal{A},\mathcal{B}}(E(u),E(u))}{E(u)}\frac{E(u)}{u}\\
&= \lim_{u\to0^+}(1-\alpha)\frac{C^{\mathcal{A},\mathcal{B}}(E(u),E(u))}{E(u)}\frac{E(u)}{u}\\
&= (1-\alpha)(a_{i1}a_{j1} + a_{i3}a_{j3})E'(0+),\\
\lambda_U(\mathrm{dGAB}_{ij}) &= \lim_{u\to0^+}\frac{2u-1+\mathrm{dGAB}_{ij}(1-u,1-u)}{u}\\
&= \lim_{u\to0^+}\frac{2u-1+\alpha C_\Sigma(D(1-u),D(1-u)) + (1-\alpha)C^{\mathcal{A},\mathcal{B}}(E(1-u),E(1-u))}{u}\\
&= \lim_{u\to0^+}(1-\alpha)\frac{2u-1+C^{\mathcal{A},\mathcal{B}}(E(1-u),E(1-u))}{u}\\
&= (1-\alpha)(a_{i1}a_{j1} + a_{i3}a_{j3})E'(1-),
\end{aligned}$$

respectively, which completes the proof. □

G. Proof of Theorem 4.2

Proof. A direct calculation shows that

$$\begin{aligned}
\mathbb{P}(\mathbf{F}\le\mathbf{u}) &= \mathbb{P}\big(I\mathbf{D}^{-1}(\mathbf{U}) + (1-I)\mathbf{E}^{-1}(\mathbf{V})\le\mathbf{u}\big)\\
&= \mathbb{E}\big[\mathbb{P}\big(I\mathbf{D}^{-1}(\mathbf{U}) + (1-I)\mathbf{E}^{-1}(\mathbf{V})\le\mathbf{u}|I\big)\big]\\
&= \textstyle\sum_{k=0,1}\mathbb{P}\big(I\mathbf{D}^{-1}(\mathbf{U}) + (1-I)\mathbf{E}^{-1}(\mathbf{V})\le\mathbf{u}|I=k\big)\cdot\mathbb{P}(I=k)\\
&= \mathbb{P}\big(\mathbf{D}^{-1}(\mathbf{U})\le\mathbf{u}\big)\cdot\mathbb{P}(I=1) + \mathbb{P}\big(\mathbf{E}^{-1}(\mathbf{V})\le\mathbf{u}\big)\cdot\mathbb{P}(I=0)\\
&= \alpha C_1(\mathbf{D}(\mathbf{u})) + (1-\alpha)C_2(\mathbf{E}(\mathbf{u})),
\end{aligned}$$

where the second equality holds due to the law of total expectation, and the fourth equality holds due to the independence between U_i, V_i, and I for $i = 1, 2, \dots, n$. □

H.. Proof of Proposition 4.1

Proof. When $I=0$, it is straightforward to see that

$$\begin{aligned}
F_j(u,i,v,\mathbf{x}) &= \mathbb{P}\Big(E_j^{-1}\Big(W1_{A_j^+} + V_j1_{A_j^\perp} + (1-W)1_{A_j^-}\Big)\le u, I=0, W\le v, \mathbf{X}\le\mathbf{x}\Big)\\
&= \mathbb{P}\Big(W1_{A_j^+} + V_j1_{A_j^\perp} + (1-W)1_{A_j^-}\le E_j(u), W\le v\Big)\cdot\mathbb{P}(I=0)\cdot\mathbb{P}(\mathbf{X}\le\mathbf{x})\\
&= (1-\alpha)\big(a_{j1}M(E_i(u),v) + a_{j2}\textstyle\prod(E_j(u),v) + W(E_j(u),v)\big)\mathbf{\Phi}_{I_n}(\mathbf{x}),
\end{aligned}$$

where the second equality holds due the independence of W, I, and V_j.

On the other hand when $I=1$, we have

$$\begin{aligned}
F_j(u,i,v,\mathbf{x}) &= \mathbb{P}\left(D_i^{-1}\left(\Phi\left(\sum_{k=1}^n a_{jk}X_k\right)\right)\le u, I=1, W\le v, \mathbf{X}\le\mathbf{x}\right)\\
&= \mathbb{P}\left(D_i^{-1}\left(\Phi\left(\sum_{k=1}^n a_{jk}X_k\right)\right)\le u, \mathbf{X}\le\mathbf{x}\right)\cdot\mathbb{P}(I=1)\cdot\mathbb{P}(W\le v)\\
&= \alpha v\mathbf{\Phi}_{\Sigma_j}\left(\Phi^{-1}\left(D_j(u)\right), \mathbf{x}\right),
\end{aligned}$$

where the second equality holds due to the independence of W, I, and $\mathbf{X}$, and the third equality holds due to the normal dependence of $\sum_{k=1}^n a_{jk}X_k$ and $\{X_k\}_{k=1}^n$. □

I. Proof of Theorem 4.3

Proof. Under Assumption G-A, the law of total probability indicates that the joint distribution of $\mathbf{U} = (U_1, U_2, \dots, U_n)$ can be expressed as

$$\mathbb{P}(\mathbf{U} \le \mathbf{u}) = \mathbb{E}\left[\mathbb{P}\left(\mathbf{U} \le \mathbf{u} | W, I, \{X_k\}_{k=1}^n\right)\right] = \mathbb{E}\left[\prod_{j=1}^n \mathbb{P}\left(\mathbf{U} \le \mathbf{u} | W, I, \{X_k\}_{k=1}^n\right)\right],$$

where $\mathbf{u} = (u_1, u_2, ..., u_n) \in [0, 1]^n$.

For notation convenience, we denote $X = \mathbb{E}\left[\prod_{j=1}^n \mathbb{P}\left(U_j \le u_j | W, I = 1, \{X_k\}_{k=1}^n\right)\right]$ and $Y = \mathbb{E}\left[\prod_{j=1}^n \mathbb{P}\left(U_j \le u_j | W, I = 0, \{X_k\}_{k=1}^n\right)\right]$. Consequently, Assumption G-B implies that

$$\mathbb{E}\left[\prod_{j=1}^n \mathbb{P}\left(\mathbf{U} \le \mathbf{u} | W, I, \{X_k\}_{k=1}^n\right)\right] = \alpha \mathbb{E}[X] + (1 - \alpha)\mathbb{E}[Y]. \tag{I.1}$$

In particular,

$$\mathbb{E}[X] = \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{1}{\alpha} \frac{\partial^{n+1} F_j(u_j, 1, v, \mathbf{x})}{\partial v \partial \mathbf{x}} \, \mathrm{d}\mathbf{x} = C_\Sigma(\mathbf{D}(\mathbf{u})), \tag{I.2}$$

and

$$\begin{aligned}
\mathbb{E}[Y] &= \int_{[0,1]} \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{1}{1-\alpha} \frac{\partial^{n+1} F_j(u_j, 0, v, \mathbf{x})}{\partial v \partial \mathbf{x}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}v = \int_{[0,1]} \prod_{j=1}^n \frac{1}{(1-\alpha)\mathbf{\Phi}_{I_n}(\mathbf{x})} \frac{\partial F_j(u_j, 0, v, \mathbf{x})}{\partial v} \, \mathrm{d}v \\
&= \int_{[0,1]} \prod_{j=1}^n \frac{\partial \left(a_{j1} M\left(E_j(u_j), v\right) + a_{j2} \prod \left(E_j(u_j), v\right) + a_{j3} W\left(E_j(u_j), v\right) \right)}{\partial v} \, \mathrm{d}v \\
&= \int_{[0,1]} \prod_{j=1}^n \left(a_{j1} 1_{\{E_j(u_j) > v\}} + a_{j2} E_j(u_j) + a_{j3} 1_{\{E_j(u_j) + v - 1 > 0\}} \right) \, \mathrm{d}v \\
&= \sum_{j_1=1}^3 \cdots \sum_{j_n=1}^3 \int_{[0,1]} \prod_{j_i=1, i \le n} \left(a_{ij_i} 1_{\{E_i(u_i) > v\}} \right) \prod_{j_i=3, i \le n} \left(a_{ij_i} 1_{\{v > 1 - E_i(u_i)\}} \right) \prod_{j_i=2, i \le n} \left(a_{ij_i} E_i(u_i) \right) \, \mathrm{d}v \\
&= \sum_{j_1=1}^3 \cdots \sum_{j_n=1}^3 \left(\prod_{i=1}^n a_{ij_i} \right) W\left(\min_{j_i=1, i \le n} \{E_i(u_i)\}, \min_{j_i=3, i \le n} \{E_i(u_i)\} \right) \prod_{j_i=2, i \le n} E_i(u_i).
\end{aligned} \tag{I.3}$$

Substituting Equations (I.2) and (I.3) into Equation (I.1) yields the dGAB copula with distortion functions $\mathbf{D} = (D_1, D_2, ..., D_n)$ and $\mathbf{E} = (E_1, E_2, ..., E_n)$. □

J the BDS pricing algorithms

In this section, we provide two algorithms for the BDS pricing. Specifically, Algorithm 2 below is based on the Monte Carlo simulation, and Algorithm 3 is to employ the closed-form solution (19) directly.

The detailed pricing procedures are summarized in the following two algorithms.

Algorithm 2 BDS pricing based on Monte Carlo simulation

1: Initializing α, d_{ij}^+, and d_{ij}^- for all $i \ne j$, e.g., $\alpha \leftarrow 0.8, d_{ij}^+ \leftarrow 0.5$, and $d_{ij}^- \leftarrow 0.5$.
2: Obtaining a_{ij} by solving equations $a_{i1}a_{j1} + a_{i3}a_{j3} = d_{ij}^+, a_{i1}a_{j3} + a_{i3}a_{j1} = d_{ij}^-$, and $\sum_{j=1}^3 a_{ij} = 1$.
3: Calibrating ρ_{ij} according to the Spearman's rank correlation coefficient ρ_{ij}^S by solving the equation: $12\int\int_{[0,1]^2} \mathrm{dGF}_{ij}(u, v) \, \mathrm{d}u\mathrm{d}v - 3 = \rho_{ij}^S$.
4: Cholesky decomposition: $\Sigma = (\rho_{ij})_{ij} = AA^\top$, where we denote $A = (\delta_{ij})_{ij}$.
5: Generating random numbers for each U_i in Equation (12).
6: Generating random numbers for $\tau_i = -\log(1 - U_i)/h_i$, where h_i is the hazard rate.
7: Averaging $1_{\{N(t)=k\}}$ across all samples to obtain the probability $\mathbb{P}(N(t) = k)$.
8: Computing the BDS price by Equation (16).
The next algorithm is to employ the closed-form solution (19) directly.

Algorithm 3 BDS pricing based on closed-form solutions in Theorem 5.1

1: Initializing α, d_{ij}^+, and d_{ij}^- for all $i \ne j$, e.g., $\alpha \leftarrow 0.8, d_{ij}^+ \leftarrow 0.5$, and $d_{ij}^- \leftarrow 0.5$.
2: Obtaining a_{ij} by solving equations $a_{i1}a_{j1} + a_{i3}a_{j3} = d_{ij}^+, a_{i1}a_{j3} + a_{i3}a_{j1} = d_{ij}^-$, and $\sum_{j=1}^3 a_{ij} = 1$.
3: Calibrating ρ_{ij} according to the Spearman's rank correlation coefficient ρ_{ij}^S by solving the equation: $12\int\int_{[0,1]^2} \mathrm{dGF}_{ij}(u, v) \, \mathrm{d}u\mathrm{d}v - 3 = \rho_{ij}^S$.
4: Obtaining ζ_i by solving the system of equations: $\zeta_i\zeta_j = \rho_{ij}$ for all $i \ne j$.
5: Computing the probbility $\mathbb{P}(N(t) = k)$ by Theorem 5.1.
6: Computing the BDS price by Equation (19).

K. Proof of Proposition 5.1

Proof. By definition of the default time τ_i, it is straightforward to see that when $I=1$, we have

$$
\begin{aligned}
p_t^{i|X,I,W} &= \mathbb{P}\left(I\cdot D_i^{-1}(Z_i)+(1-I)\cdot E_i^{-1}(W_i)\le t|X,I=1,W\right)\\
&= \mathbb{P}\left(F_i^{-1}\left(D_i^{-1}\left(\Phi\left(\zeta_i X_i+\sqrt{1-\zeta_i^2}X\right)\right)\right)\le t|X,I=1,W\right)\\
&= \mathbb{P}\left(X_i\le \frac{\Phi^{-1}(D_i(F_i(t)))-\sqrt{1-\zeta_i^2}X}{\zeta_i}|X,I=1,W\right)\\
&= \Phi\left(\frac{\Phi^{-1}(D_i(F_i(t)))-\sqrt{1-\zeta_i^2}X}{\zeta_i}\right).
\end{aligned}
$$

On the other hand, when $I=0$, we have

$$
\begin{aligned}
p_t^{i|X,I,W} &= \mathbb{P}\left(ID_i^{-1}(Z_i)+(1-I)E_i^{-1}(W_i)\le t|X,I=0,W\right)\\
&= \mathbb{P}\left(W1_{A_i^+}+V_i1_{A_i^\perp}+(1-W)1_{A_i^-}\le E_i(F_i(t))|X,I=0,W\right)\\
&= \textstyle\sum_{A=A_i^+,A_i^\perp,A_i^-}\mathbb{P}\left(W1_{A_i^+}+V_i1_{A_i^\perp}+(1-W)1_{A_i^-}\le E_i(F_i(t)),1_A=1|X,I=0,W\right)\\
&= a_{i1}1_{\{W\le E_i(F_i(t))\}}+a_{i2}E_i(F_i(t))+a_{i3}1_{\{W\ge 1-E_i(F_i(t))\}},
\end{aligned}
$$

where the third equality holds due to the fact that $\left(A_i^+,A_i^\perp,A_i^-\right)$ is a random partition of the probability space. □

L. Proof of Theorem 5.1

Proof. By Equation (16), it suffices to compute the probability $\mathbb{P}(N(t)=k)$ under the default times specified by Equation (17). The moment generating function of $N(t)$ is employed, which is defined as

$$
\psi_{N(t)}(u)=\mathbb{E}[u^{N(t)}]=\sum_{k=0}^{n}\mathbb{P}(N(t)=k)u^k,\ u\in[0,1]. \tag{L.1}
$$

By letting $N_i(t)=1_{\{\tau_i\le t\}}$, it follows that $N(t)=\sum_{i=1}^n N_i(t)$. Applying the law of total expectation to Equation (L.1), it follows that

$$
\psi_{N(t)}(u)=\mathbb{E}_{X,I,W}\left[\mathbb{E}\left[u^{N(t)}|X,I,W\right]\right]=\mathbb{E}_{X,I,W}\left[\Pi_{i=1}^n\mathbb{E}\left[u^{N_i(t)}|X,I,W\right]\right], \tag{L.2}
$$

where the second equality holds due to the conditional independence of $\{N_i(t)\}_{i=1}^n$ given three common factors (X, I, W). Specifically, the conditional expectation can be evaluated explicitly as

$$
\begin{aligned}
\mathbb{E}\left[u^{N_i(t)}|X,I,W\right] &= \mathbb{P}\left(N_i(t)=0|X,I,W\right)+\mathbb{P}\left(N_i(t)=1|X,I,W\right)u\\
&= 1-p_t^{i|X,I,W}+p_t^{i|X,I,W}u.
\end{aligned}
$$

Substituting it into Equation (L.2) yields

$$
\begin{aligned}
\psi_{N(t)}(u) &= \mathbb{E}_{X,I,W}\left[\Pi_{i=1}^n\left(1-p_t^{i|X,I,W}+p_t^{i|X,I,W}u\right)\right]\\
&= \alpha\mathbb{E}_{X,W}\left[\Pi_{i=1}^n\left(1-p_t^{i|X}+p_t^{i|X}u\right)\right]+(1-\alpha)\mathbb{E}_{X,W}\left[\Pi_{i=1}^n\left(1-q_t^{i|W}+q_t^{i|W}u\right)\right]\\
&= \alpha\textstyle\int_{-\infty}^{\infty}\Pi_{i=1}^n\left(1-p_t^{i|x}+p_t^{i|x}u\right)\phi(x)\ \mathrm{d}x+(1-\alpha)\int_0^1\Pi_{i=1}^n\left(1-q_t^{i|w}+q_t^{i|w}u\right)\ \mathrm{d}w,
\end{aligned} \tag{L.3}
$$

where ϕ is the density function of standard normal distributions.

By comparing the coefficients of the term u^k for $k=1,2,\dots,n$ in Equations (L.1) and (L.3), we can obtain the key probability $\mathbb{P}(N(t)=k)$ for the BDS pricing. Specifically, by direct calculation, the coefficient of u^k in Equation (L.3) is

$$
\alpha\int_{-\infty}^{\infty}\sum_{|I|=k}\prod_{i\in I}p_t^{i|x}\prod_{u\notin I}(1-p_t^{i|x})\phi(x)\,\mathrm{d}x+(1-\alpha)\int_0^1\sum_{|I|=k}\prod_{i\in I}q_t^{i|w}\prod_{u\notin I}(1-q_t^{i|w})\mathrm{d}w,
$$

where I is a subset of $\{1,2,\dots,n\}$ and $|I|$ denotes the cardinality of the set I. A rearrangement of the above equation gives

$$
\mathbb{P}(N(t)=k)=\sum_{|I|=k}\left(\alpha\int_{-\infty}^{\infty}\prod_{i\in I}p_t^{i|x}\prod_{u\notin I}\left(1-p_t^{i|x}\right)\phi(x)\ \mathrm{d}x+(1-\alpha)\int_0^1\prod_{i\in I}q_t^{i|w}\prod_{u\notin I}\left(1-q_t^{i|w}\right)\ \mathrm{d}w\right).
$$

Substituting it into the pricing formula (16) yields the closed-form solution to the BDS price under the dGAB copula, namely Equation (19), which completes the proof. □

M. The dGAB-EM algorithm

In this appendix, we detail the dGAB-EM algorithm.

M.1. dGAB-E step

For the dGAB-E step in Algorithm 1, we have

$$w_j^t = \frac{p\left(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}_{iter}\right) \cdot p(z_t = j; \boldsymbol{\theta}_{iter})}{\sum_{k=1}^4 p\left(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}_{iter}\right) \cdot p(z_t = j; \boldsymbol{\theta}_{iter})} = \frac{c_{\rho_{iter}^{(j)}}\left(\mathbf{D}_j(\mathbf{u}_t)\right) \cdot \alpha_{iter}^{(j)}}{\sum_{j=1}^4 c_{\rho_{iter}^{(j)}}\left(\mathbf{D}_j(\mathbf{u})_t\right) \cdot \alpha_{iter}^{(j)}}, \tag{M.1}$$

where $\rho_{iter}^{(2)} = 0.99, \rho_{iter}^{(3)} = 0$, and $\rho_{iter}^{(4)} = -0.99$ for all integers $iter \geq 0$. The dGAB-EM algorithm updates the parameter $\boldsymbol{\theta} = \left(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, \rho\right)$ util it converges.

M.2. dGAB-M step

For the dGAB-M step, we have the maximum likelihood function below:

$$l(\boldsymbol{\theta}) = \sum_{t=1}^n \sum_{j=1}^4 w_j^t \log \frac{p\left(\mathbf{u}_t | z_t = j; \boldsymbol{\theta}\right) \cdot p(z_t = j; \boldsymbol{\theta})}{w_j^t} = \sum_{t=1}^n \sum_{j=1}^4 w_j^t \log \frac{c_{\rho^{(j)}}\left(\mathbf{D}_j(\mathbf{u}_t)\right)\alpha^{(j)}}{w_j^t},$$

where $\boldsymbol{\theta} = \left(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, \rho\right)$ and $\mathbf{u}_t = \left(u_t^{(1)}, u_t^{(2)}\right)$. The maximizer $\hat{\theta}$ of the above likelihood function can be found by the first-order conditions.

In particular, when all distortion functions are trivial, namely in the case of GAB copula, the process of optimizing the likelihood function becomes more straightforward. To find the correlation coefficient $\hat{\rho}$, it is equivalent to maximizing the objective function $\mathcal{J}(\boldsymbol{\theta}) = \sum_{t=1}^n w_1^t \log c_\rho(\mathbf{u}_t)$, which can be done by the first-order condition directly. To find the optimal $\alpha^{(i)}$, we apply the Lagrange multiplier due to the constraint $\sum_{j=1}^4 \alpha^{(i)} = 1$. Namely, we have the following objective function:

$$\mathcal{L}(\boldsymbol{\theta}) = l(\boldsymbol{\theta}) + \lambda\left(\sum_{j=1}^4 \alpha^{(j)} - 1\right).$$

Taking the first derivative yields

$$\frac{\mathrm{d}\mathcal{L}(\boldsymbol{\theta})}{\mathrm{d}\alpha^{(j)}} = \sum_{t=1}^n \frac{w_j^t}{\alpha^{(j)}} + \lambda, \quad \text{for } j = 1, 2, 3, 4.$$

Consequently, we obtain the closed-form solution to the parameter $\alpha^{(j)}$, which takes the following form:

$$\alpha^{(j)} = \frac{1}{n}\sum_{t=1}^n w_j^t = \frac{1}{n}\sum_{t=1}^n \frac{c_{\rho_{iter}^{(j)}}(\mathbf{u}_t) \cdot \alpha_{iter}^{(j)}}{\sum_{k=1}^4 c_{\rho_{iter}^{(k)}}(\mathbf{u}_t) \cdot \alpha_{iter}^{(k)}}, \quad \text{for } j = 1, 2, 3, 4. \tag{M.2}$$

Combining Equations (M.1) and (M.2) completes the dGAB-EM algorithm.

N. Error analysis in the approximation method

Lemma N.1. *For any $(u, v) \in (0, 1)^2$, we have $\lim_{\rho \to 1-} C_\rho(u, v) = M(u, v)$, $\lim_{\rho \to 0} C_\rho(u, v) = \Pi(u, v)$, and $\lim_{\rho \to -1+} C_\rho(u, v) = W(u, v)$.*

Proof. By the definition of the Gaussian copula, it is straightforward to verify that $\lim_{\rho \to 0} C_\rho(u, v) = \Pi(u, v)$, thus it suffices to prove the cases when $\rho \to 1-$ and $-1+$.

Let X and Y be two independent standard normal random variables, and define $X_1 = X$ and $X_2 = \rho X + \sqrt{1 - \rho^2} Y$ for $\rho > 0$. It is obvious that for any $(a_1, a_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
\Phi_\rho(a_1, a_2) &= \mathbb{P}(X_1 \leq a_1, X_2 \leq a_2) = \mathbb{P}\left(X \leq a_1, X \leq \frac{a_2 - \sqrt{1 - \rho^2} Y}{\rho}\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(X \leq a_1, X \leq \frac{a_2 - \sqrt{1 - \rho^2} Y}{\rho}\right) \middle| Y\right] \\
&= \int_{\mathbb{R}} \Phi\left(\min\left(a_1, \frac{a_2 - \sqrt{1 - \rho^2} y}{\rho}\right)\right) \mathrm{d}\Phi(y) \\
&= \int_{-\infty}^{\frac{a_2 - \rho a_1}{\sqrt{1-\rho^2}}} \Phi(a_1) \, \mathrm{d}\Phi(y) + \int_{\frac{a_2 - \rho a_1}{\sqrt{1-\rho^2}}}^{+\infty} \Phi\left(\frac{a_2 - \sqrt{1 - \rho^2} y}{\rho}\right) \mathrm{d}\Phi(y).
\end{aligned} \tag{N.1}$$

Without losing generality, we assume that $0 \le u \le v \le 1$. Noting the definition of the Gaussian copula $C_\rho(u,v) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$, the difference between $C_\rho(u,v)$ and $M(u, v)$ can be evaluated as

$$
\begin{aligned}
|M(u,v) - C_\rho(u,v)| &= \left| \int_{\frac{\Phi^{-1}(v)-\rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}}^{+\infty} \left(u - \Phi\left(\frac{\Phi^{-1}(v) - \sqrt{1-\rho^2}y}{\rho} \right) \right) \, \mathrm{d}\Phi(y) \right| \\
&= \left| u \left(1 - \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}} \right) \right) - \int_{\frac{\Phi^{-1}(v)-\rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}}^{+\infty} \Phi\left(\frac{\Phi^{-1}(v) - \sqrt{1-\rho^2}y}{\rho} \right) \, \mathrm{d}\Phi(y) \right| \qquad \text{(N.2)} \\
&\le 2\left(1 - \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}} \right) \right) = 2\left(1 - \Phi\left(\frac{\Phi^{-1}(v) - \Phi^{-1}(u)}{\sqrt{1-\rho^2}} + \Phi^{-1}(u)\sqrt{\frac{1-\rho}{1+\rho}} \right) \right),
\end{aligned}
$$

where the first equality holds due to Equation (N.1).

The inequality (N.2) indicates that $\lim_{\rho \to 1-} C_\rho(u,v) = M(u,v)$ for $0 \le u \le v \le 1$. The case $0 \le v \le u \le 1$ is the same and the proof is omitted. In addition to the case when $\rho \to 1-$, the case when $\rho \to -1+$, i.e., $\lim_{\rho \to -1+} C_\rho(u,v) = W(u,v)$ can be proved similarly. □

The lemma provides insight into the behavior of the Gaussian copula in the limit as ρ approaches 1, 0, and -1, where it converges to the comonotonic, independent, and countermonotonic copulas, respectively. In the specific context of the dGAB-EM algorithm, it is essential to acknowledge that the dataset's capacity is finite, i.e., the dataset length $n = 965 < +\infty$. Therefore, the convergence can be extended to the "uniform convergence", i.e., for any $\epsilon > 0$, there exist ρ_+ and ρ_- such that

$$
\begin{aligned}
\left| M(u,v) - C_\rho(u,v) \right| < \epsilon, &\quad \text{if } \rho \in (\rho_+, 1), \\
\left| W(u,v) - C_\rho(u,v) \right| < \epsilon, &\quad \text{if } \rho \in (-1, \rho_-),
\end{aligned}
$$

hold for all (u, v) belonging to the dataset of finite capacity.

During the execution of the dGAB-EM algorithm, it is feasible to select values for ρ_+ and ρ_- in proximity to 1 and -1, respectively. Consequently, as we estimate the parameters within the approximated model denoted as C_{apx} in Equation (22), the parameters will ultimately converge toward their actual values. This convergence is facilitated by the continuity of all relevant functions and the finite nature of the iterative process. The empirical results substantiating this convergence are presented in the following table.

As evidenced by the table, the estimated parameter $\hat{\theta} = \left(\widehat{\alpha^{(1)}}, \widehat{\alpha^{(2)}}, \widehat{\alpha^{(3)}}, \widehat{\alpha^{(4)}}, \hat{\rho} \right)$ converges as ρ_+ and ρ_- converge to 1 and -1, respectively.